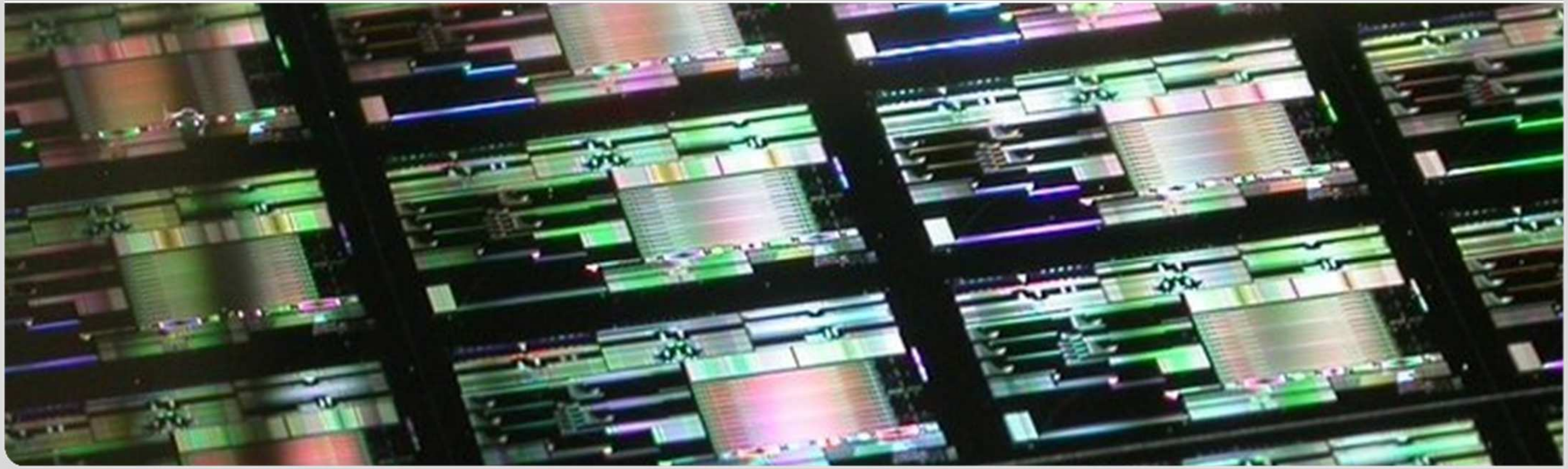
Optical Waveguides and Fibers

Christian Koos

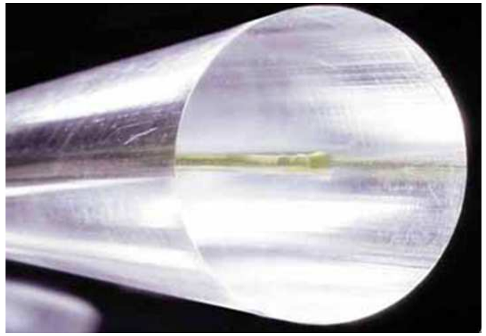
Institute of Photonics and Quantum Electronics



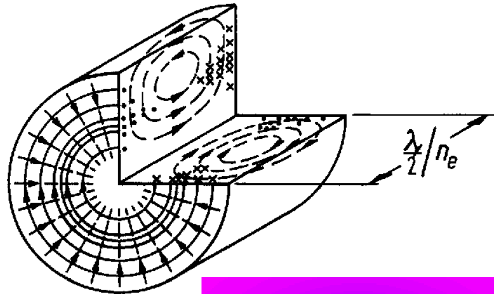
Lecture 1

Optical Waveguides and Fibers

Lecture: Prof. Dr. Christian Koos
Tue, 15:45 - 17:15 h
First lecture: Kl. ETI, Bldg. 11.10
Further lectures: IPQ meeting room (R. 3.42), Bldg. 30.10

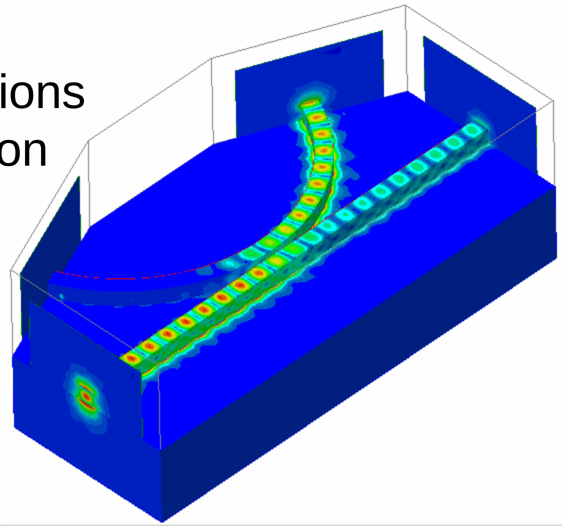


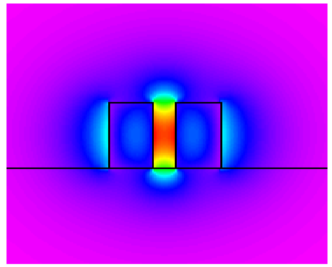
Tutorial: Dipl.-Ing. Jörg Pfeifle
M. Sc. Aleksandar Nesic
Dipl.-Phys. P. Dietrich
Wed ,11:30 - 13:00 h
IPQ meeting room (R. 3.42), Bldg. 30.10

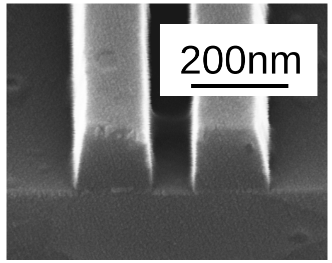


Contents:

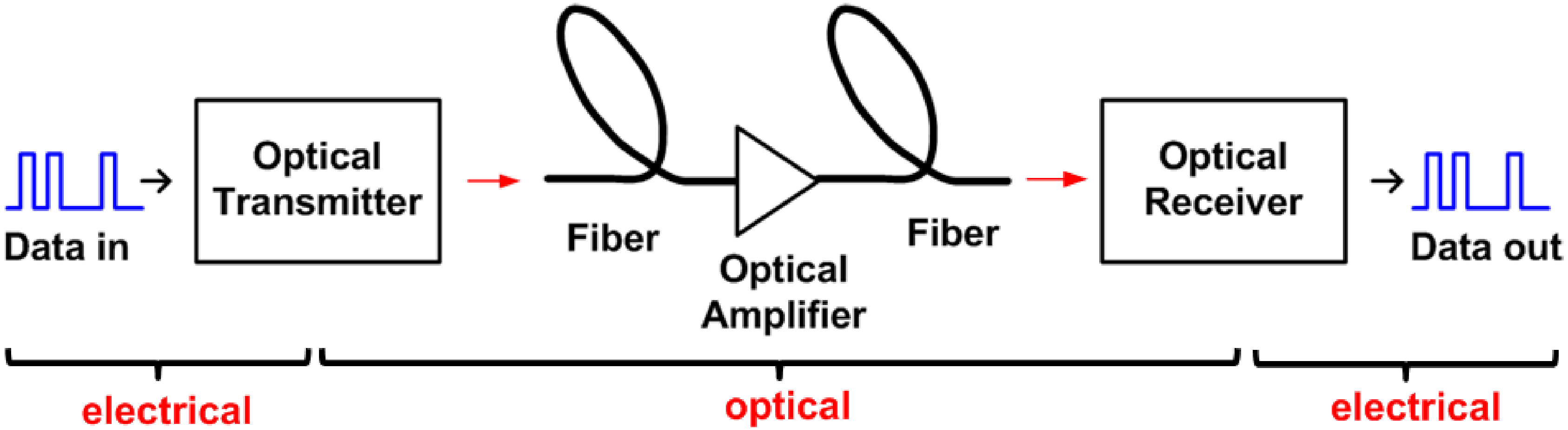
- Introduction: Optical communications
- Fundamentals of wave propagation in optics
- Slab waveguides
- Planar integrated waveguides
- Optical fibers
- Waveguide-based devices and systems







Communication with light



Advantages of Optical Communications:

- Large transmission capacity**:** Large fiber bandwidth of 250 …190 THz = 60 THz

  Bitrates of typical services:

  Voice (ISDN) 64 kbit/s (compressed < 10 kbit/s)
  Picture (TV) 140 Mbit/s (compressed 2…6 Mbit/s)

  Bitrates of transmission media:

  Twisted pair 6 Mbit/s (6 km); coax 650 Mbit/s (1.5 km)
  Glass fibre 1.28 Tbit/s single channel (240 km) HHI 2006
  Fibre + WDM > 100 Tbit/s (10 Billion ISDN, 20 Million TV)

- Long transmission distance due to low fiber loss

  Down to 0.15 dB/km @ λ = 1.55 µm; 0.35 dB/km @ λ = 1.3 µm; 2.2 dB/km @ λ = 0.85 µm, i. e., down to 3 dB (50%) power loss for a fibre length of L = 20 km

- Immunity to electromagnetic interference

  High carrier frequency and strong confinement of the light inside the waveguide

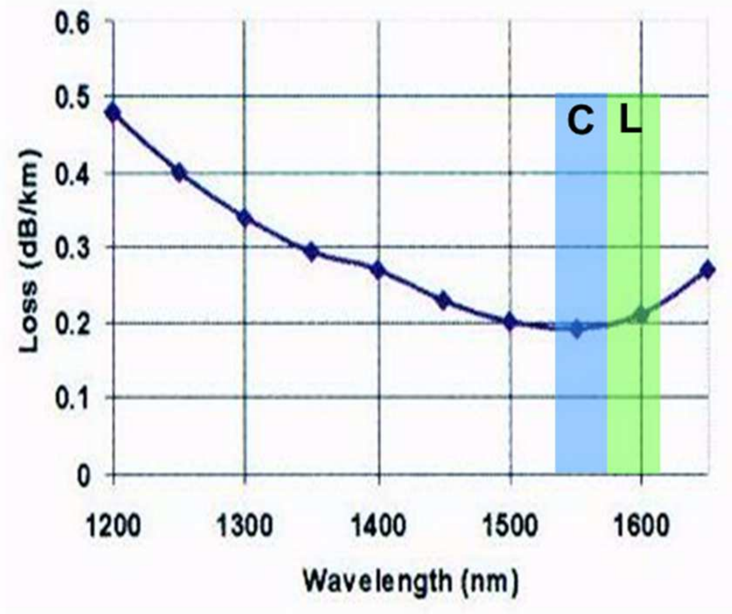
Wavelength-Division Multiplexing (WDM)

How to exploit full transmission bandwidth of optical transmission systems ?

fibres: $B \approx$ 65 THz (450 nm)

amplifiers: $B \approx$ 10 THz (80 nm)

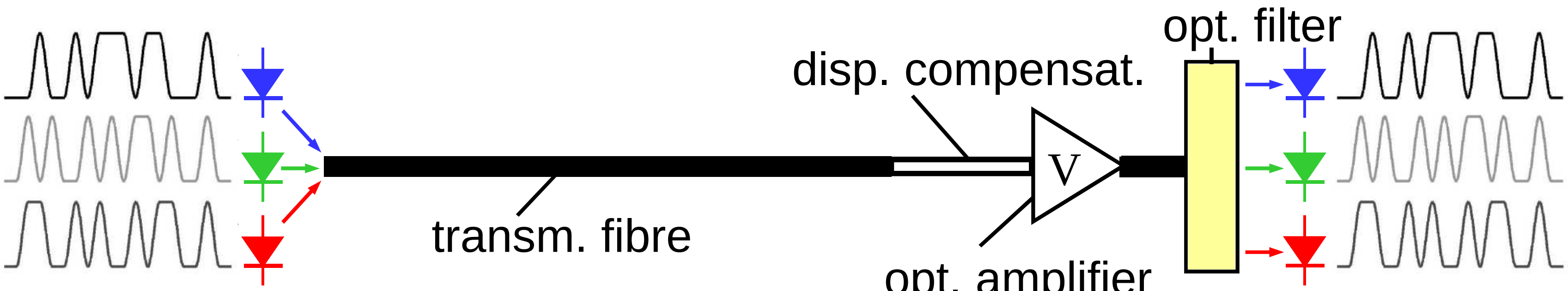
*w*avelength *d*ivision *m*ultiplexing (WDM)



channels: $\Delta f \approx$ 5, 10, 25, 50, 100 GHz

capacity: 40 Gbit/s $\times$ 100 ch = 4 Tbit/s

Can be further increased by polarization multiplexing and higher-order modulation formats.



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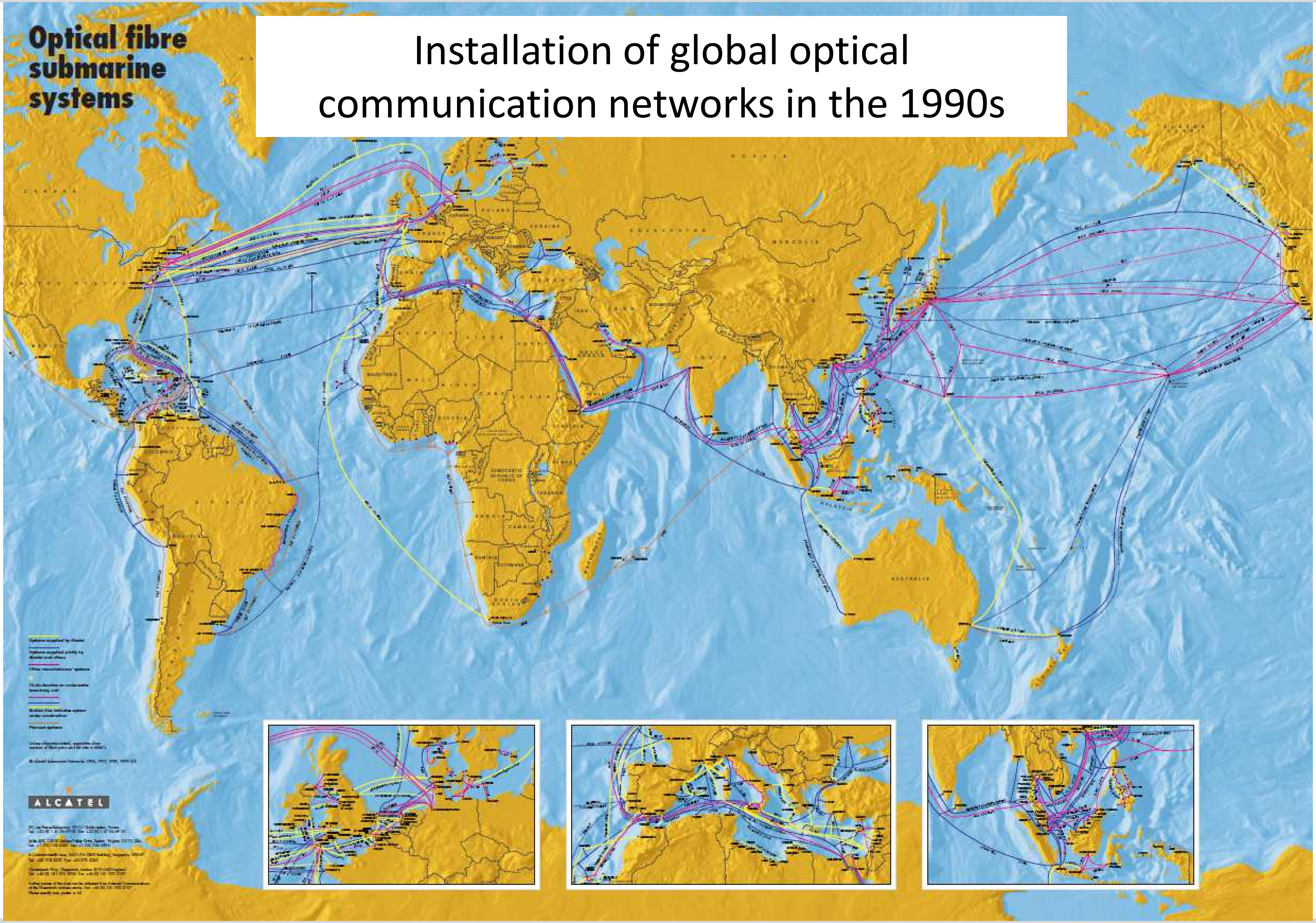
Milestones in fiber-optic communications

- **Low-loss fibers (1970's)**
  Reduction of loss from 1000 dB/km to below 20 dB/km by removing impurities, suggested in the 1960's by Charles Kao (Nobel prize 2009)
- **Semiconductor lasers operating continuously at room temperature (1980's)**
  Double heterostructure pn-junctions (first GaAs as base material, today mainly InP)
- **Erbium-doped fiber amplifier (1990's)**
  Broadband amplification (1535 - 1565 nm, 4 THz) of tens of channels in wavelength-division multiplexing (WDM) systems

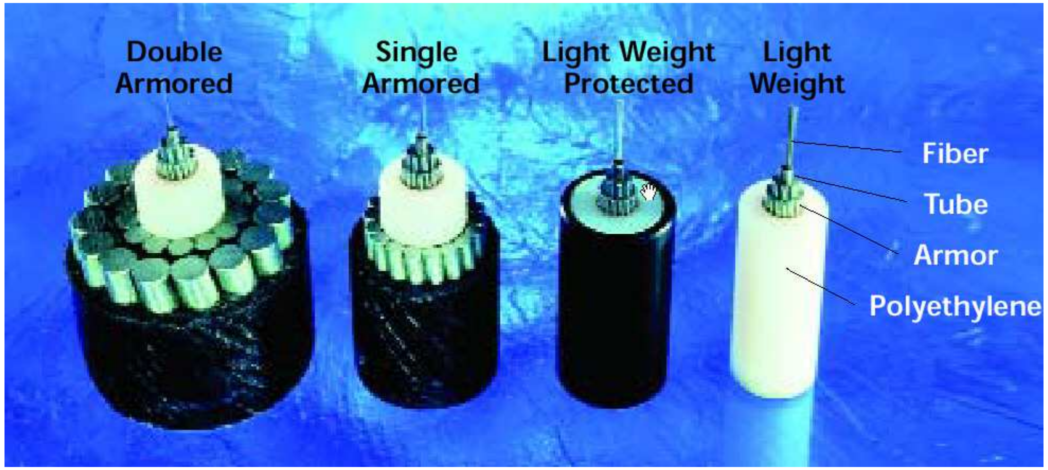
Latest developments:

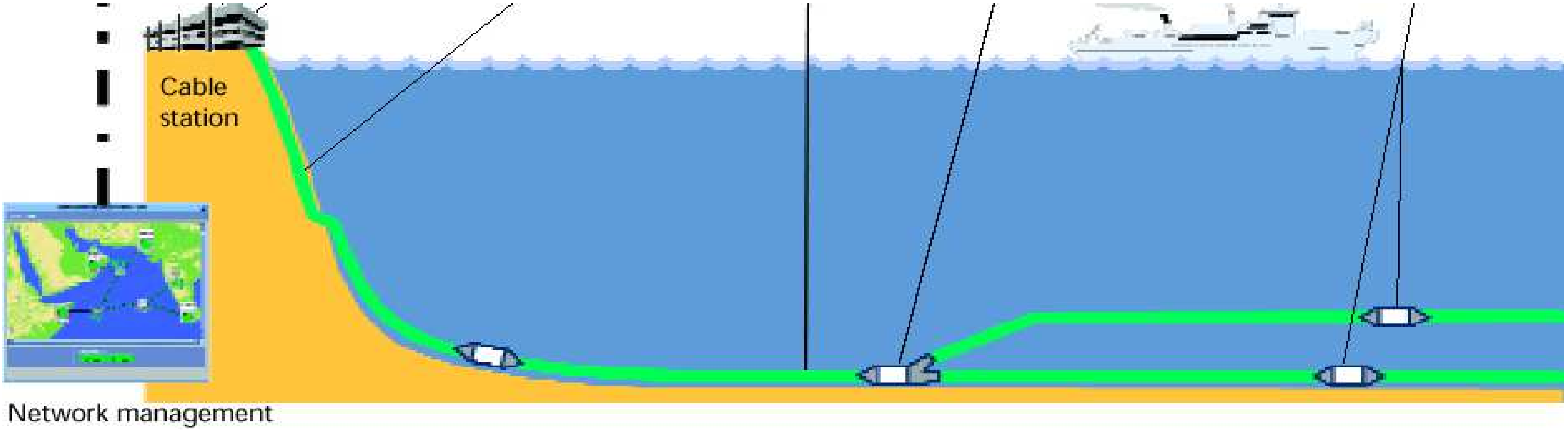
- **Coherent communications and digital signal processing**
  Encoding of data on both the amplitude and the phase of the signal by using higher-order modulation formats (e.g., quadrature amplitude modulation, QAM); compensation of transmission impairments by digital signal processing
- **Large-scale photonic integration**
  Co-integration of various different optical components (lasers, modulators, photodetectors, passive devices) on a common chip; this technology is key for realizing transmitter and receivers for coherent communications

Installation of global optical communication networks in the 1990s



Submarine Communication Systems — Laying the Cable





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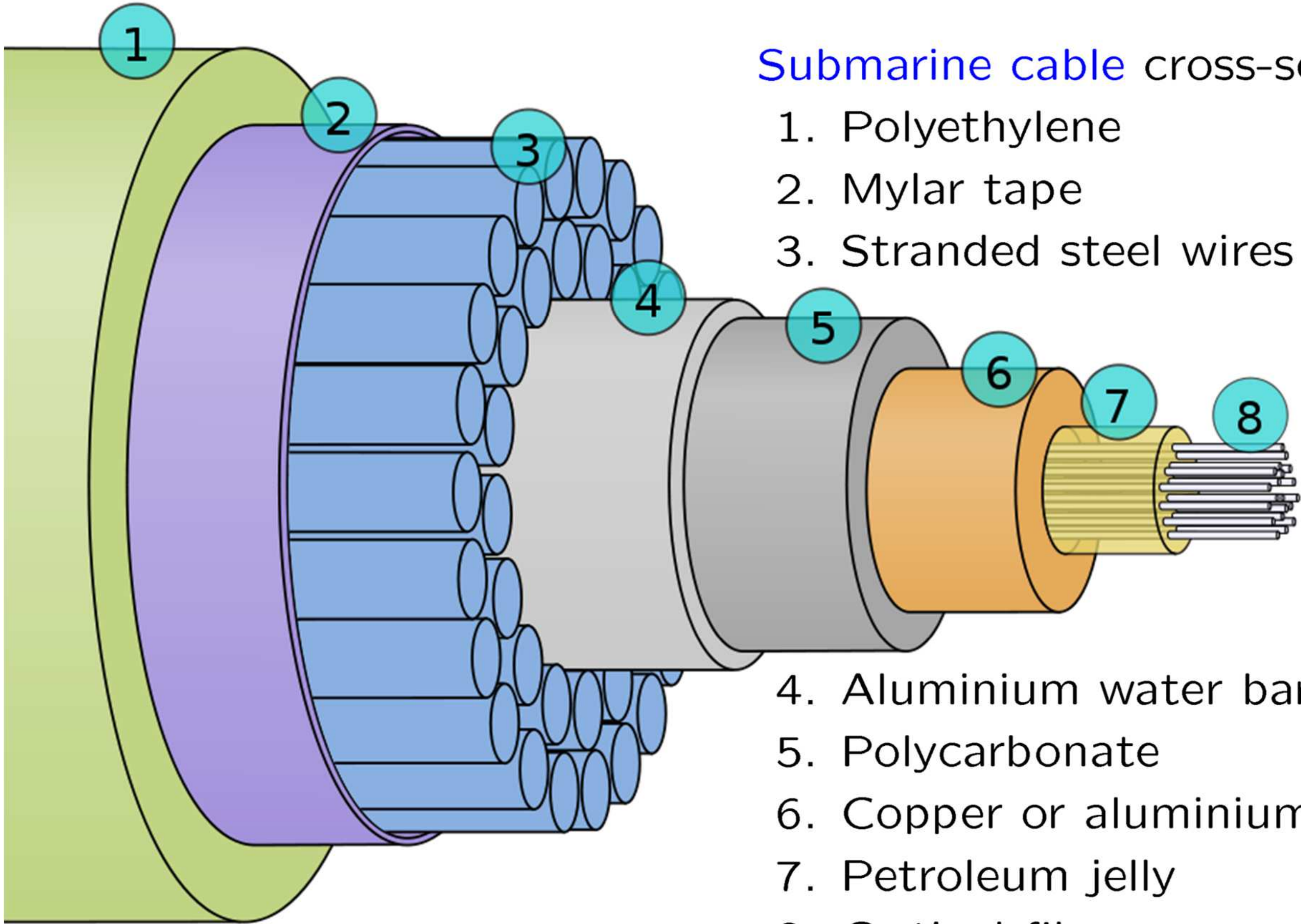
Submarine Communications Cable



http://en.wikipedia.org/wiki/Submarine_communications_cable

Submarine cable cross-section:

1. Polyethylene
2. Mylar tape
3. Stranded steel wires
4. Aluminium water barrier
5. Polycarbonate
6. Copper or aluminium tube
7. Petroleum jelly
8. Optical fibers

Data centers … and interconnect bottlenecks!

- More than 90% of all computing, storage, communication will occur within warehouse-scale data centers
- Internal data traffic much larger than access traffic

Facebook‘s data center in Luleå, Sweden: On-line since 06/2013

www.facebook.com

Large-scale data centers: The interior

Scaling of computing and storage by massive parallelization:

~ 40-80 servers per rack,
16 racks per cluster,
~ 100 clusters (100 000 servers) per data center

Problem: Network scalability is lagging behind

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Network scaling requirements

„Amdahl‘s rule of thumb“: Balanced systems for parallel computing

For every 1 MHz of "processing power"

… 1 MB of memory

… 1 Mbit/s I/O data rate

… in the late 1960's …

In 2011:

6 x 2.5 GHz processors, 2 - 4 cores each

⇒ ~ 30-60 GHz of "processing power"

24-64 GByte memory

But 1 Gbit/s of network bandwidth ???

How to …

… deliver 40 Gbit/s bandwidth to each of 100k servers ?

… deliver 40 Tbit/s to each of 100 clusters ?

… scale up to a 4 Pbit/s network?

⇒ Optical interconnects!

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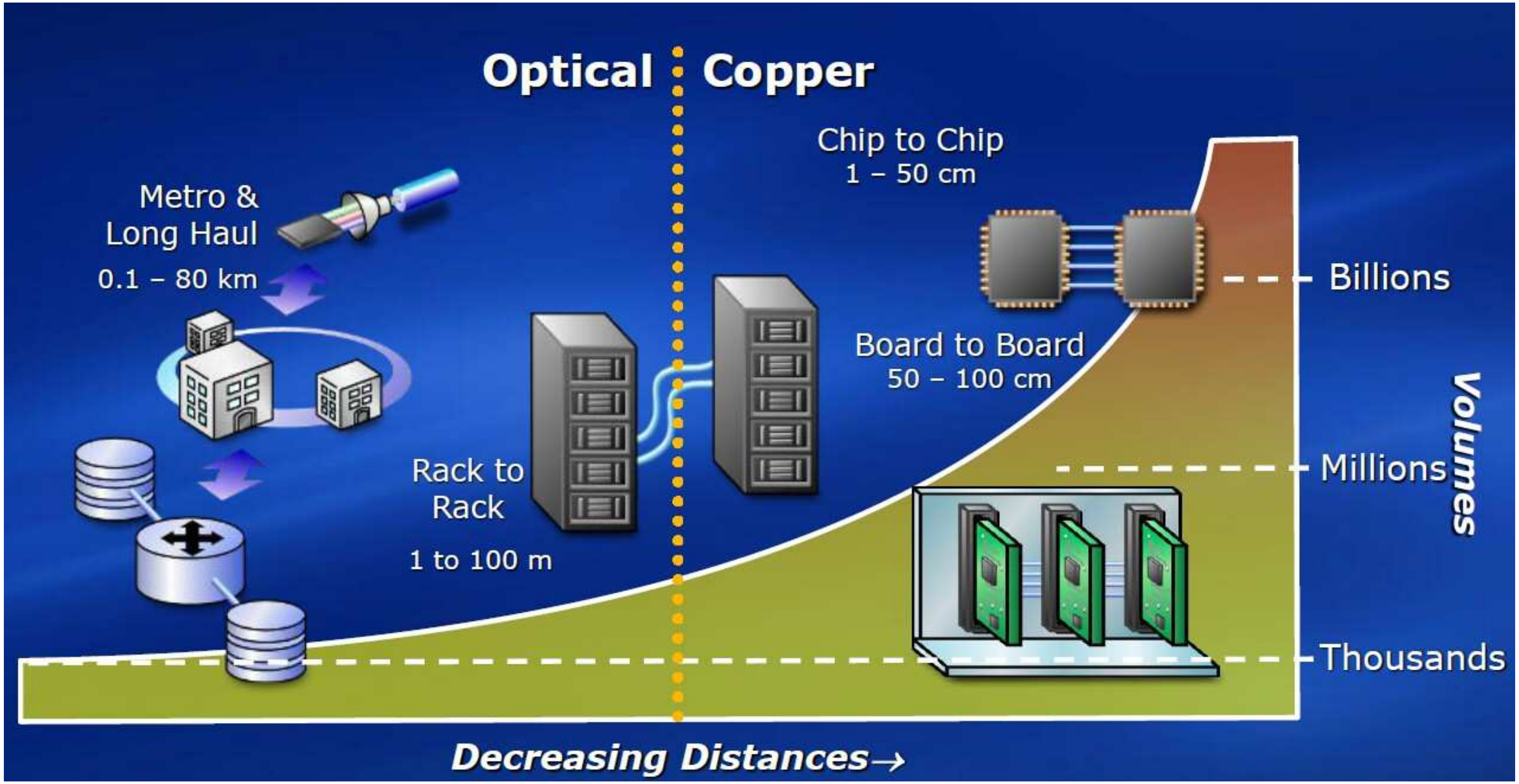
Optical interconnects

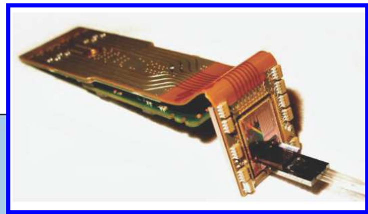
Optical communications is moving to short and medium distances

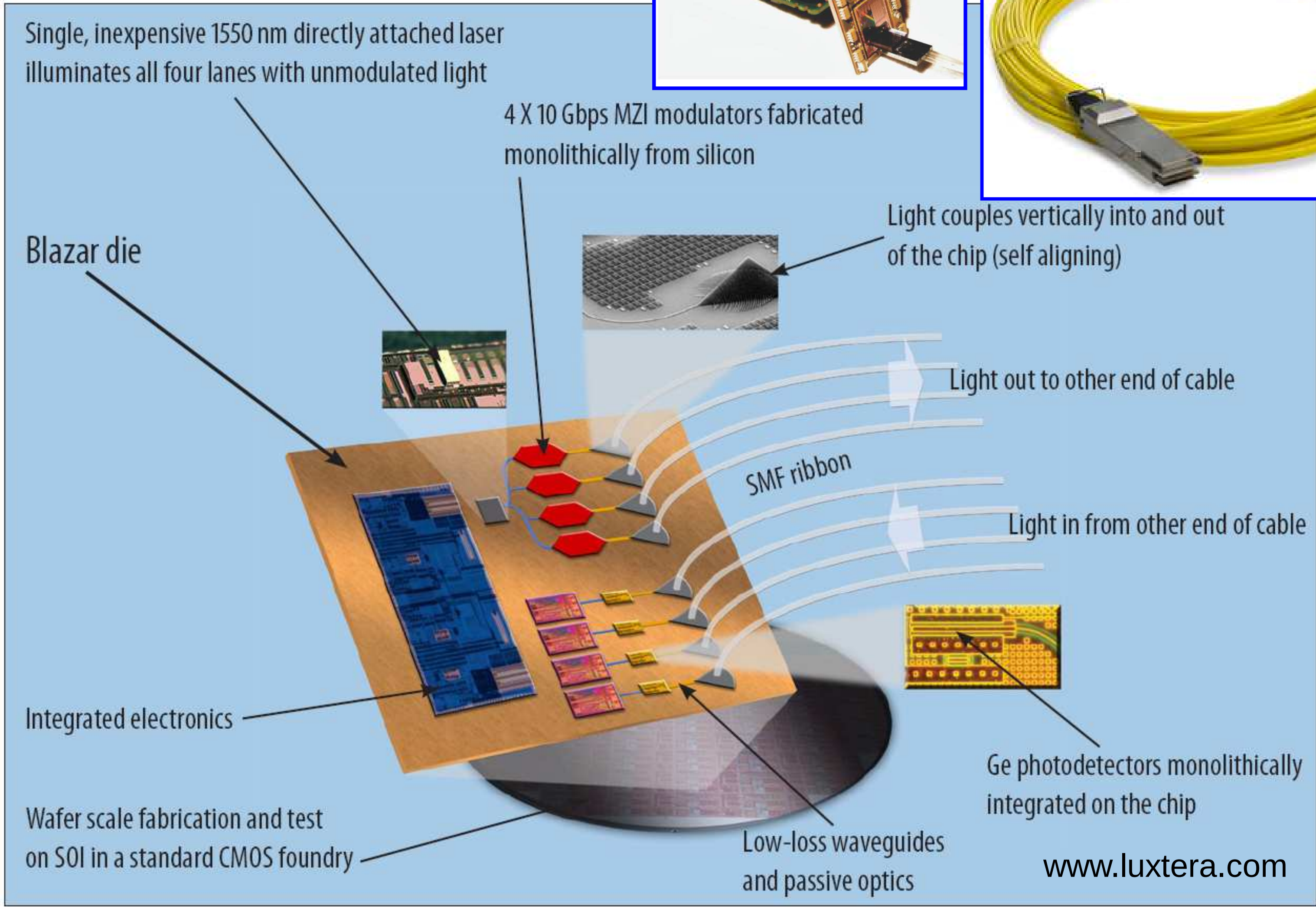


http://citrix.cleanrooms.com

Optical rack-to-rack interconnects: Active optical cables

Luxtera Blazar LUX 5010
4 x 10 Gbit/s Active Optical Cable





Optical rack-to-rack interconnects

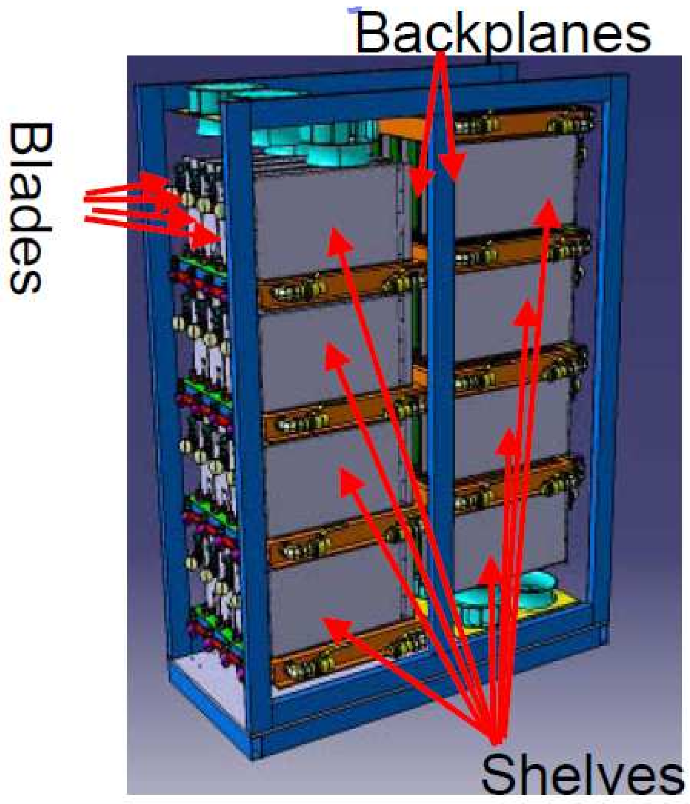
High-performance supercomputer
(IBM Roadrunner)

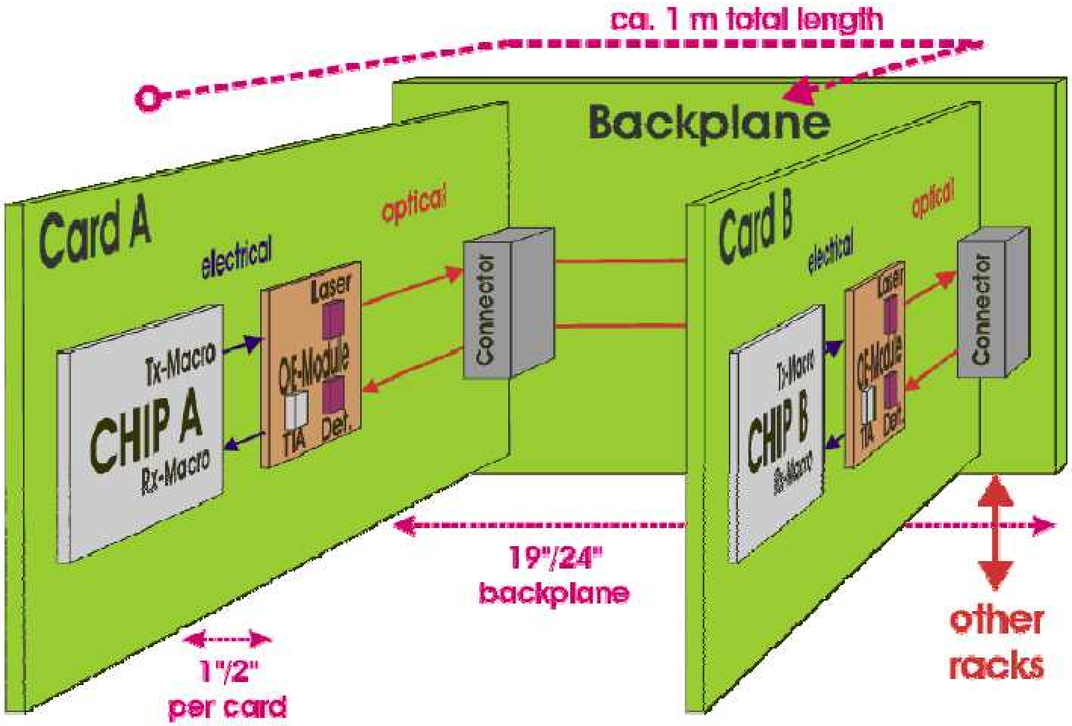
19 872 processors, 1 Pflop/s

Length:	~ 100 m
No. of links:	5 - 10 k
Bandwidth:	~ 10 Gbit/s per link
Power:	50 mW / (Gbit/s)

Optical Switch

Optical board-to-board interconnects



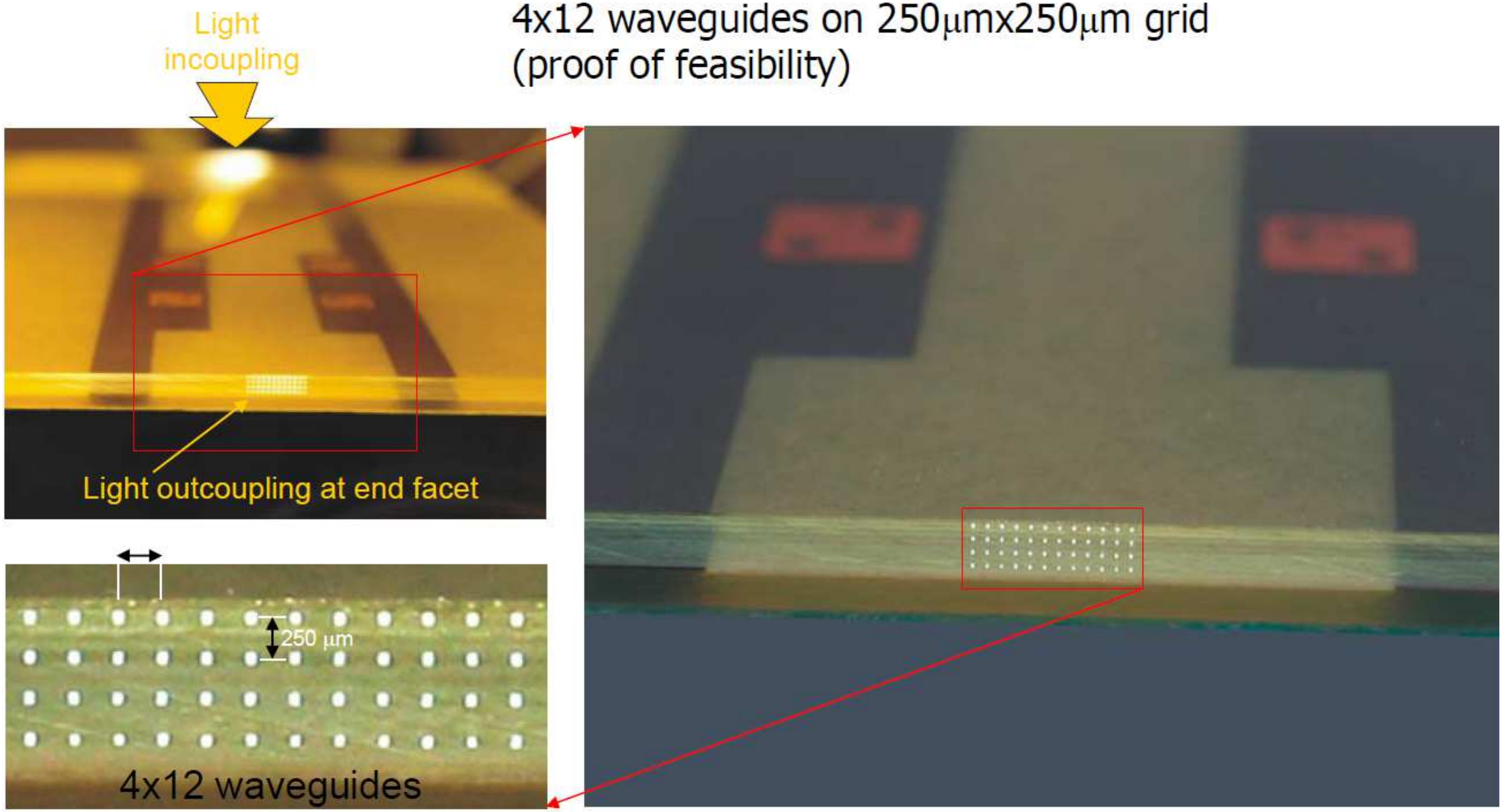


Length: ~ 50 – 100 cm
No. of links: ~ 10 k
Bandwidth: ~ 10 Gbit/s per link
Power: 10 mW / (Gbit/s)

Yurii Vlasov, 'Silicon photonics for next generation computing systems', ECOC 2008

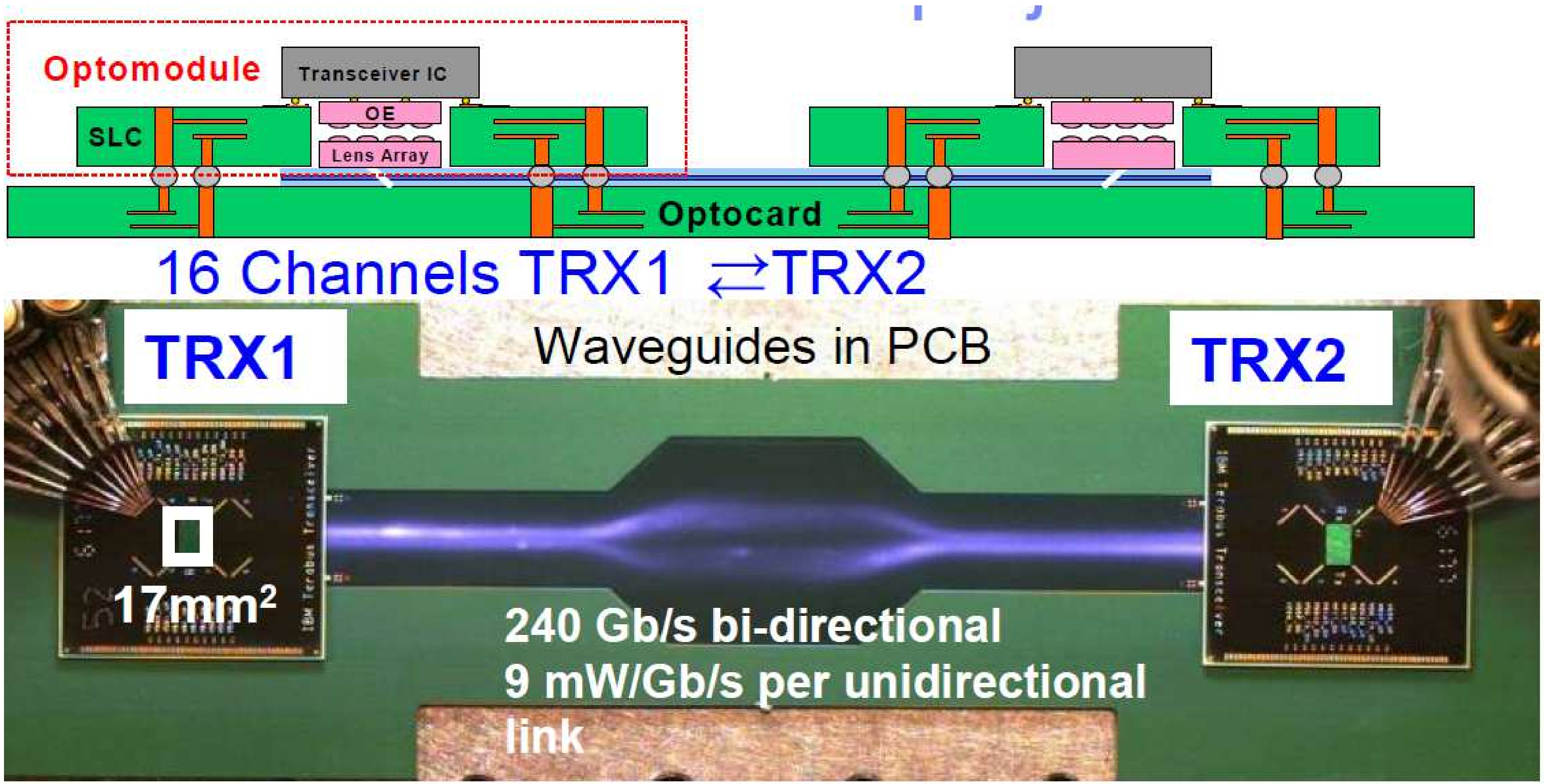
Waveguides in printed circuit boards (PCB)

4x12 waveguides on 250μmx250μm grid
(proof of feasibility)



Yurii Vlasov, 'Silicon photonics for next generation computing systems', ECOC 2008

Chip-to-chip interconnects



IBM Terabus project

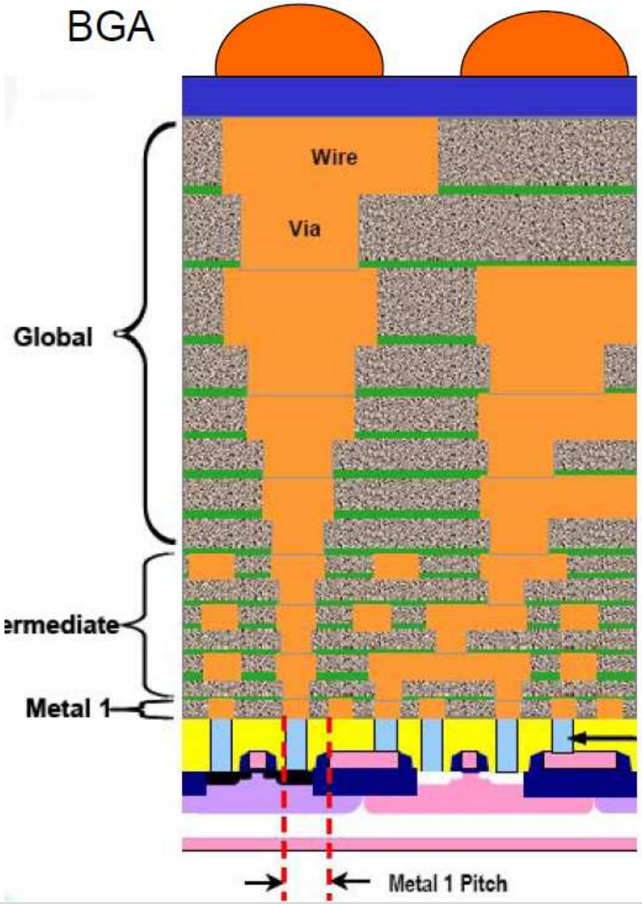
Yurii Vlasov, 'Silicon photonics for next generation computing systems', ECOC 2008

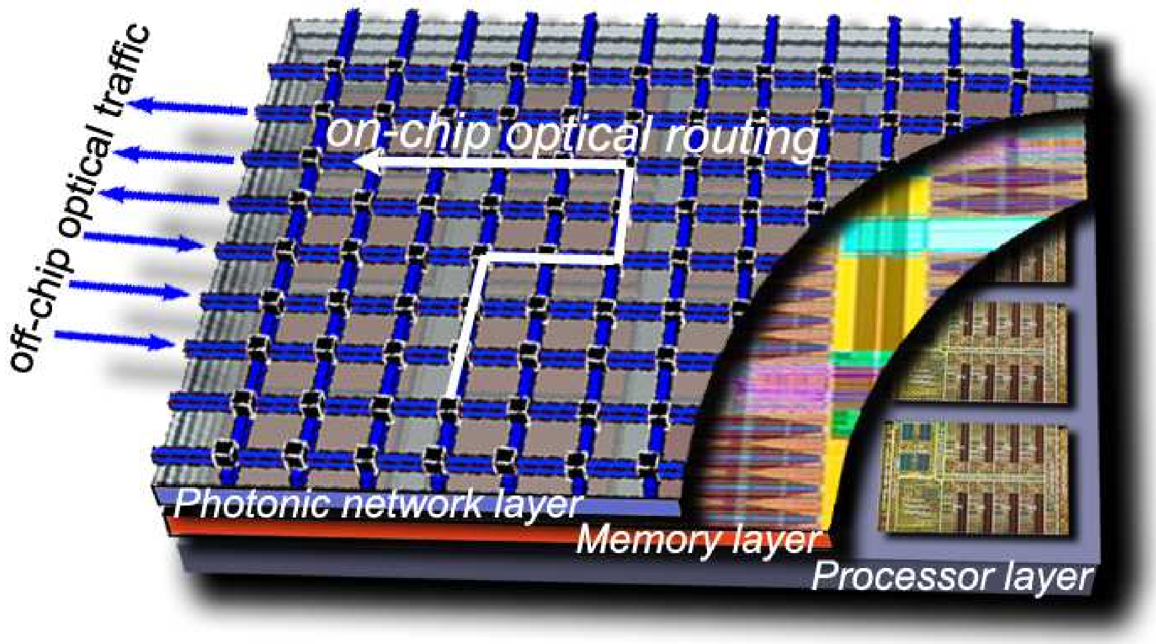
Length: ~ 1 cm
No. of links: ~ 100 k
Bandwidth: ~ 1 Tbit/s per link
Power: < 10 mW / (Gbit/s)

On-chip links

Problem today:
Performance of computers limited by bandwidth and energy consumption of electrical interconnects!

The vision: 10 Tflop/s on a 3D chip
Logic plane: ~ 300 cores
Photonic plane: ~ 70 Tbit/s on chip and off-chip, On-chip routing and switching of traffic



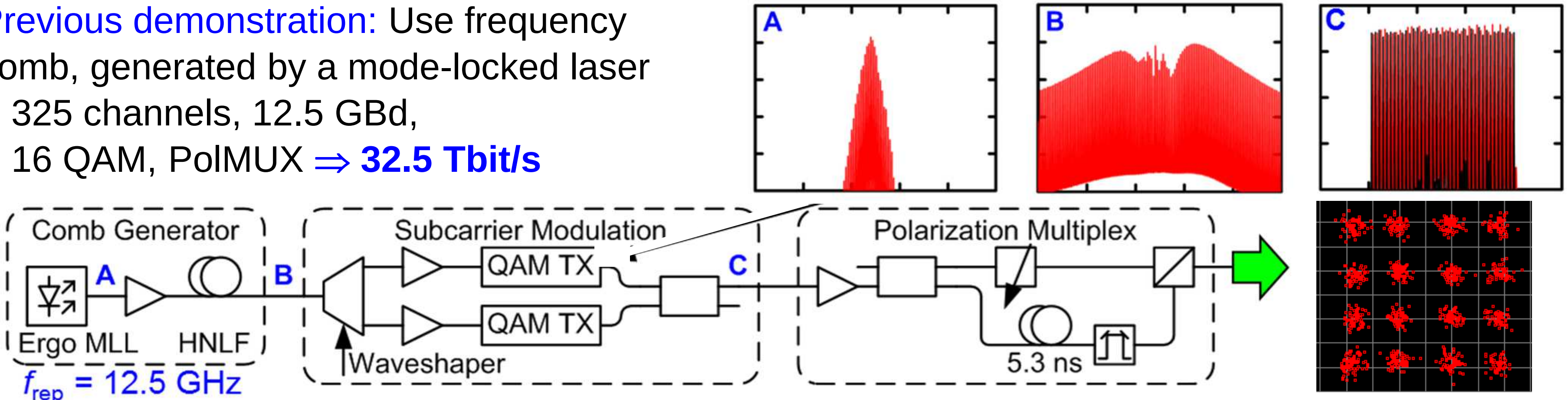


http://domino.research.ibm.com/comm/research_projects.nsf/pages/photonics.index.html

Research at IPQ – a few examples

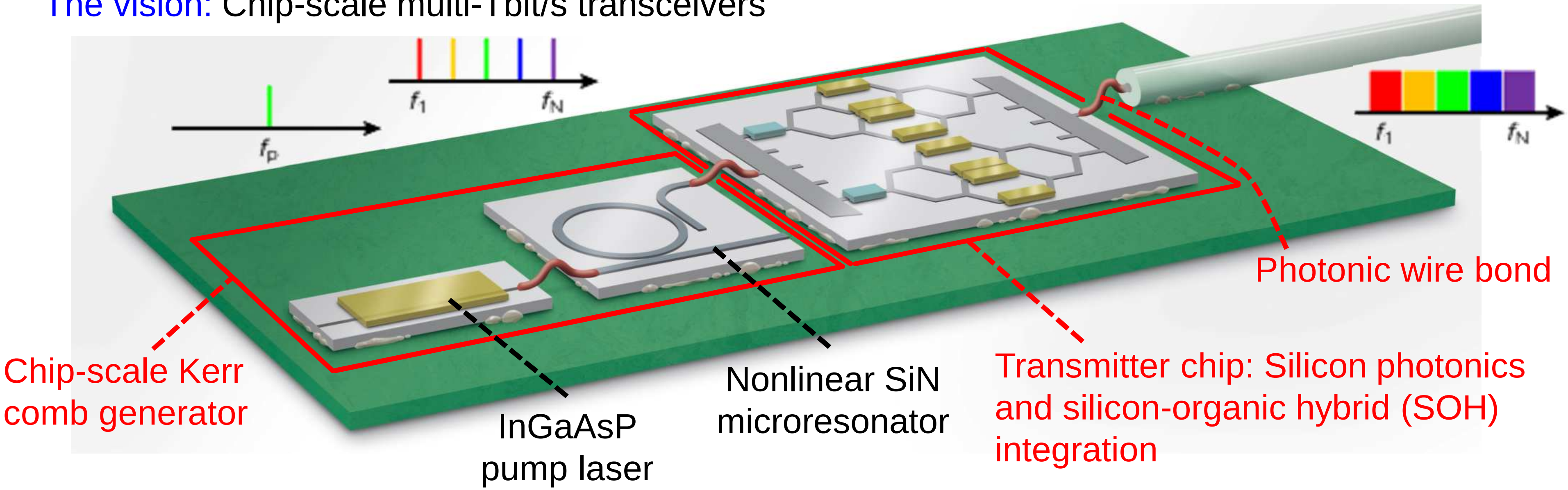
Chip-scale transceivers for massively parallel WDM

Previous demonstration: Use frequency comb, generated by a mode-locked laser
325 channels, 12.5 GBd,
16 QAM, PolMUX ⇒ **32.5 Tbit/s**



Hillerkuss *et al.*, Nat. Photon. 5, 364–371 (2011)

The vision: Chip-scale multi-Tbit/s transceivers



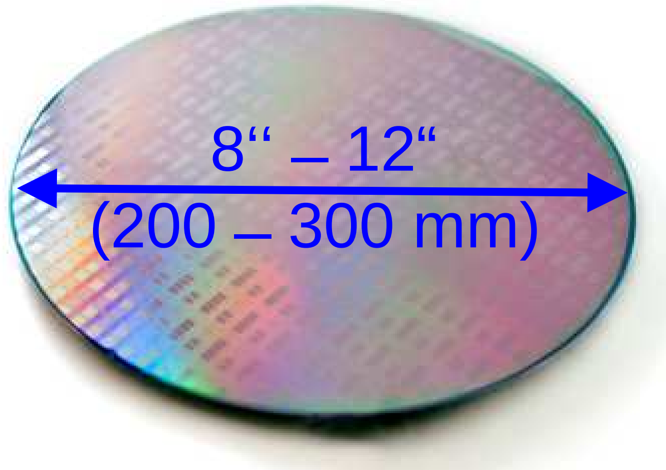
Pfeifle *et al.*, Nat. Photon. 8, 375 - 380 (2014)

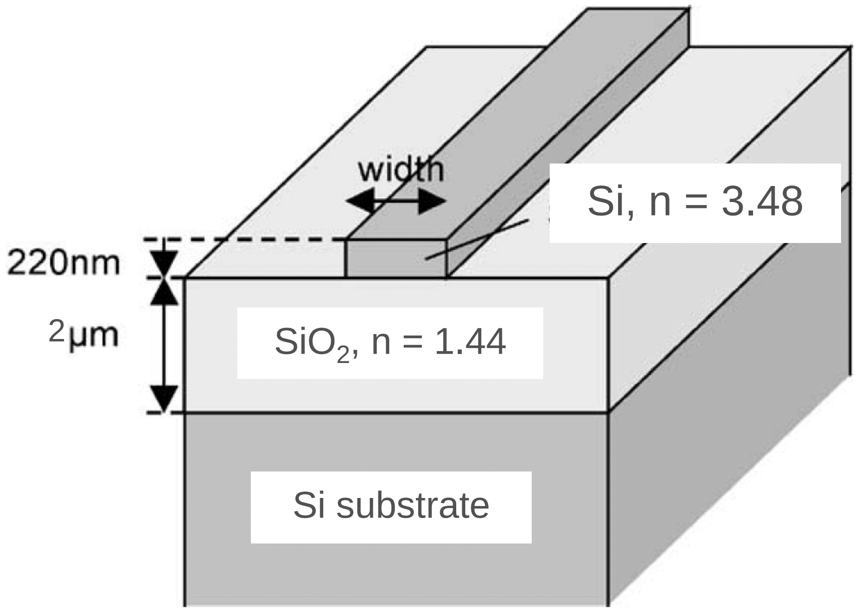
PIC design and fabless fabrication

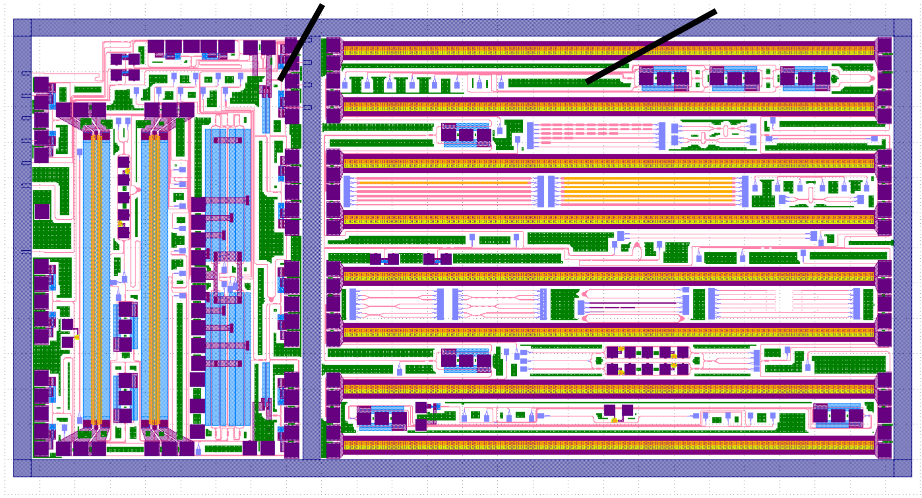
Silicon photonics:

- High-density integration by using high-index-contrast silicon-on-insulator (SOI) waveguides
- Use of CMOS foundries for photonic devices

⇒ Multi-project-wafer (MPW) shuttle runs, e.g., ePIXfab (http://www.epixfab.eu/) or



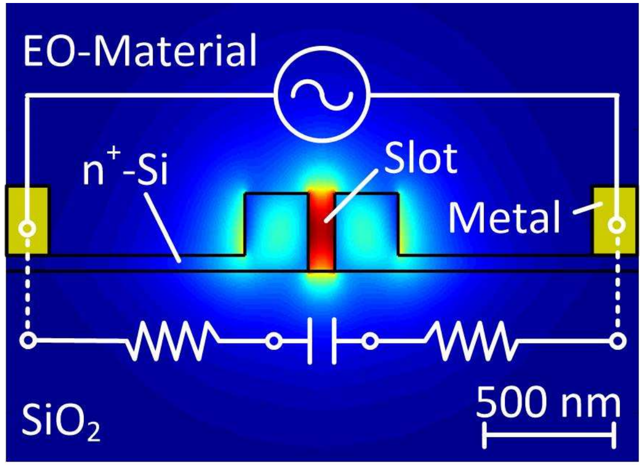


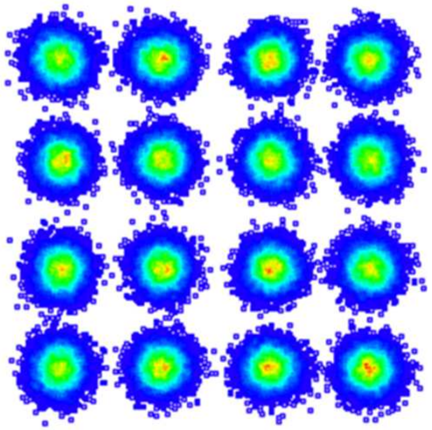


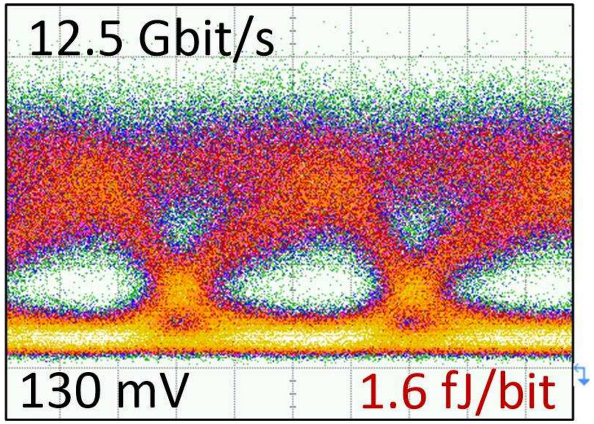
Silicon-organic hybrid (SOH) integration

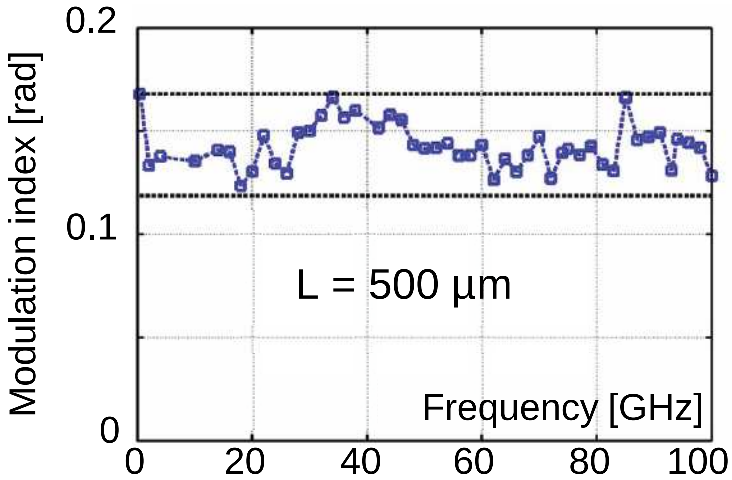
Concept: Combine nanophotonic silicon waveguides with electro-optic organic cladding materials

- High-speed modulation: 3 dB bandwidth > 100 GHz (All-silicon devices: 30 GHz)
- Highly efficient: $U_\pi L$ < 1 Vmm (All-silicon devices: $U_\pi L$ = 10 … 40 V mm)
- Lowest energy consumption of a Mach-Zehnder modulator (MZM) in any material system: < 2 fJ/bit (All-silicon MZM devices: 200 fJ/bit)
- No amplitude-phase coupling: Enables higher-order modulation formats (16 QAM)



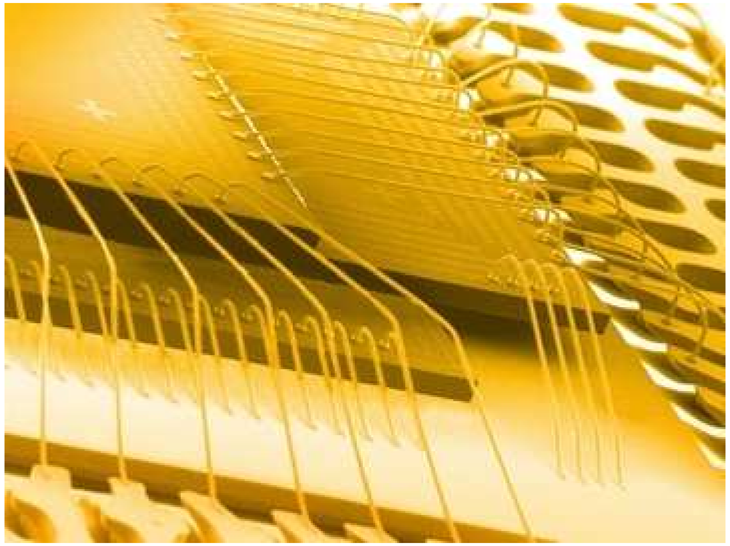






Palmer *et al.*; ECOC 2013, paper We.3.B.3 – Best student paper award
Alloatti *et al.*, Opt. Express 19 (12), 11841-11851 (2011)
Palmer *et al.*, IEEE Photonics Journal 5, 6600907 – 6600907 (2013)
Korn *et al.*, Opt. Express 21; 13219–13227 (2013)

3D photonic integration and photonic wire bonding



Electronic wire bonding: **Stacked-die package**

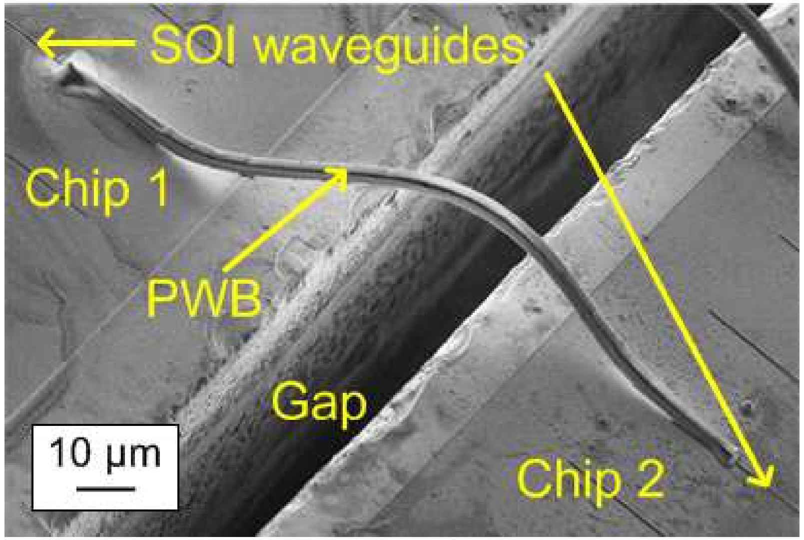
- Automated fabrication (10's of connections per second)
- Small pitch (down to 30 µm with 15 µm wire diameter)
- ± 2 µm bond placement accuracy
- Tight control of the loop trajectory

Picture source: Kulicke & Soffa, http://www.kns.com/

Photonic wire bonding: **Replace metallic wire by a 3D freeform polymer waveguide**

- No high-precision/active alignment required
- High interconnect density
- Fast fabrication

Lindenmann *et al.*, Opt. Express **20**, 17667-17677 (2012)

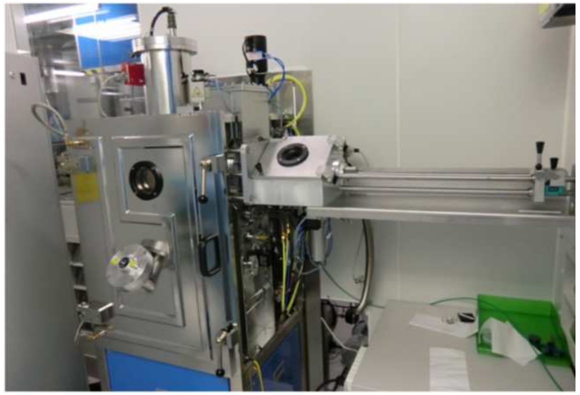


KIT Photonic Fab
@ Institute of Microstructure Technology (IMT)

- In-house fabrication processes for rapid prototyping based on electron-beam lithography
  Silicon and silicon-nitride PIC and plasmonic devices
- Technology base: Karlsruhe NanoMicro Facility (KNMF)
  Fully equipped cleanroom (500 m²) comprising state-of-the-art nanofabrication tools

Electron-beam lithography:
Vistec VB6 (100 kV)

Reactive Ion Etching:
Oxford Plasmalab System 100 with ICP 380 source



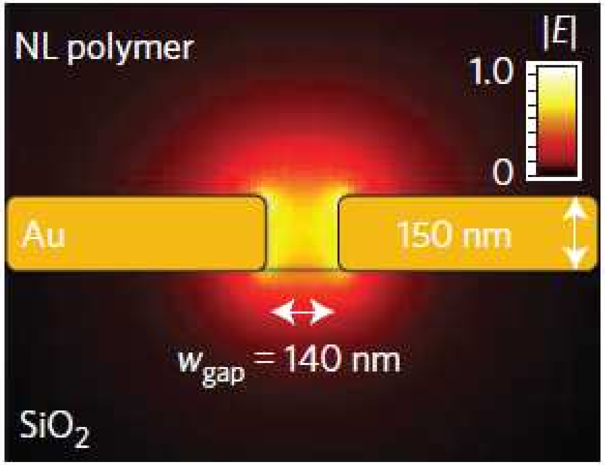
Thin-film deposition
tbd.

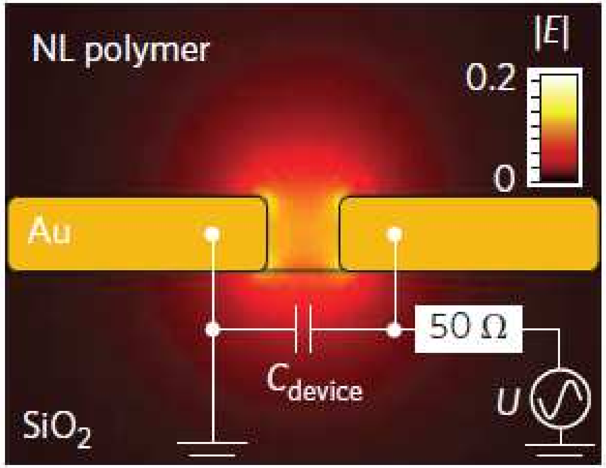
Next steps:
- Completion of available process portfolio by thin-film layer growth
- Involvement of further (IMT) personnel
- Definition and development of reproducible fabrication processes for rapid prototyping of PIC and plasmonic devices

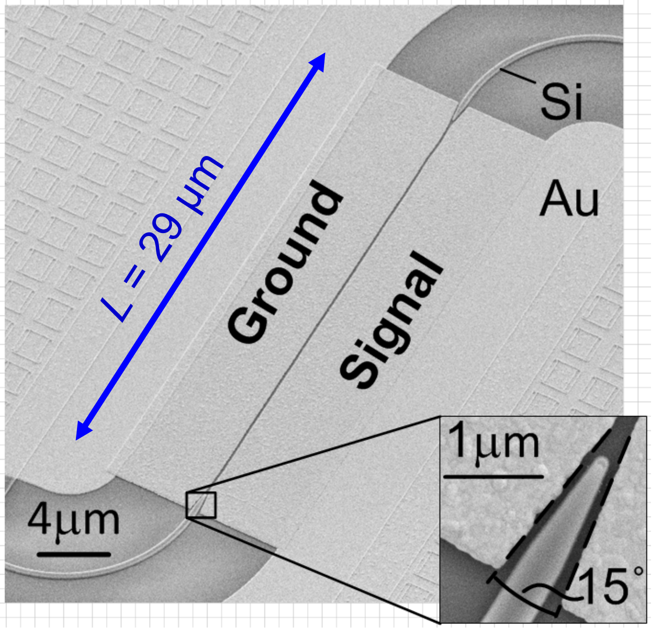
Plasmonic-organic hybrid (POH) devices

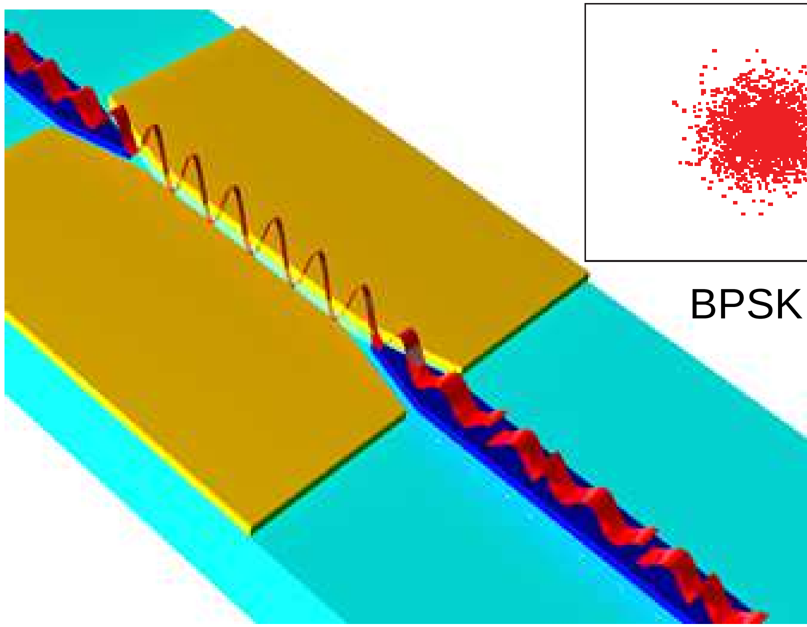
Manipulation of light on a nanometer scale

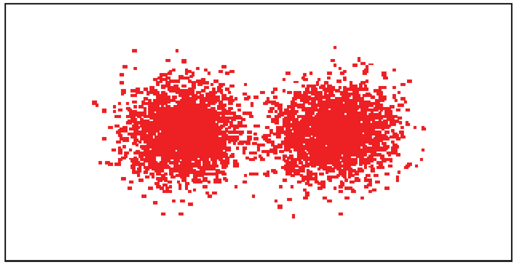
Metal slot-waveguide modulator









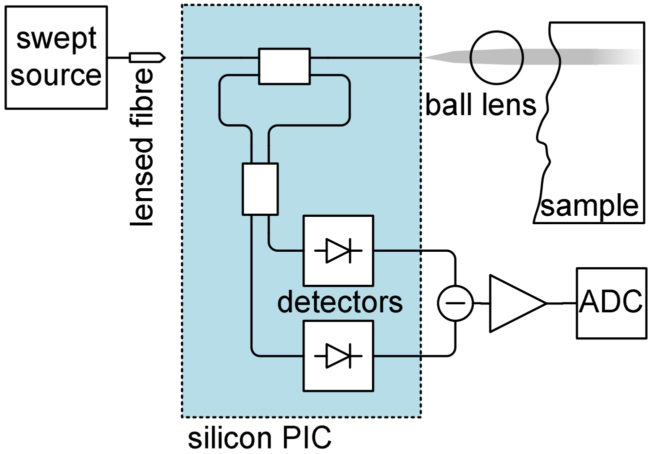


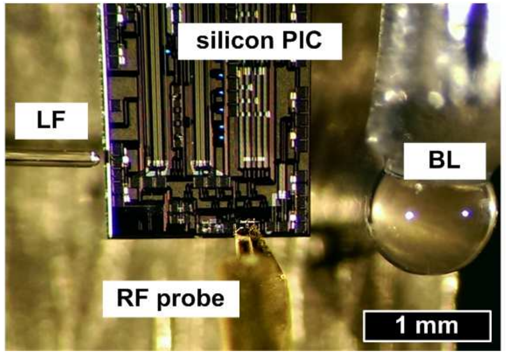
BPSK @ 40 GBd

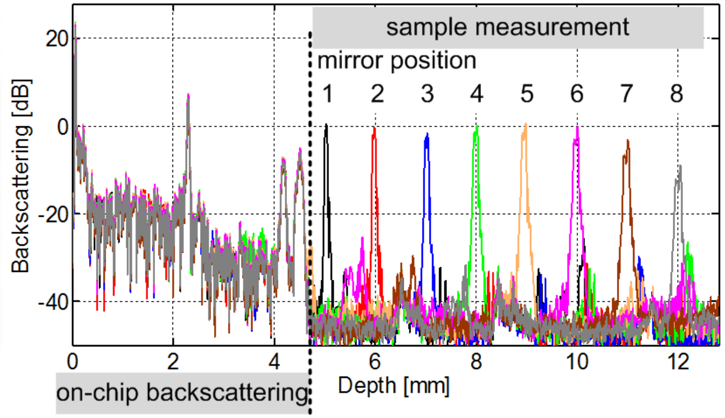
Melikyan *et al.*, Nature Photonics **8**, 229–233 (2014)

Applications in Optical Metrology

Silicon Photonic Optical Coherence Tomography



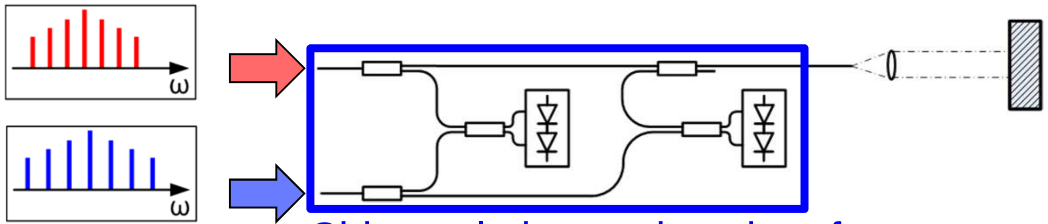




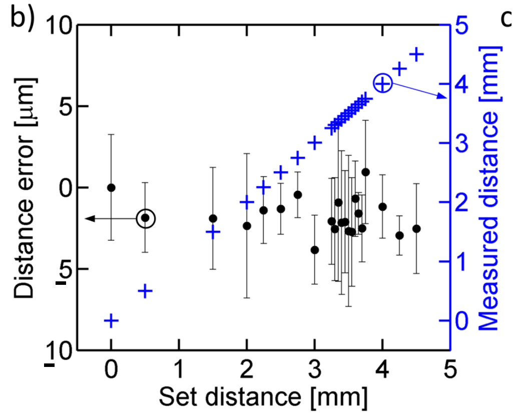
Interferometer and detectors integrated on silicon → 40 dB dynamic range over 5 mm depth

Schneider et al., CLEO Conference, San Jose, USA, ATu2P.4 (2014)

Silicon Photonic Distance Metrology



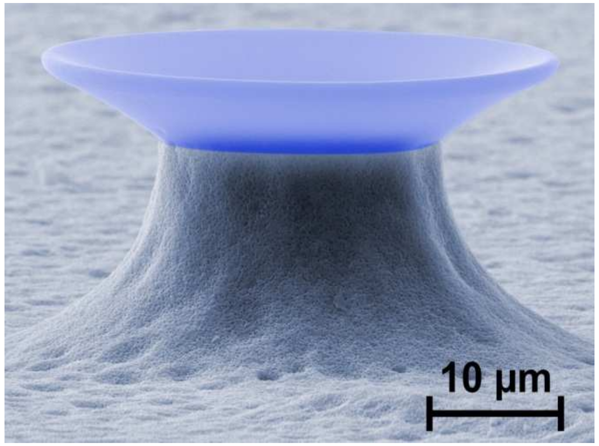
Chip-scale heterodyne interferometer



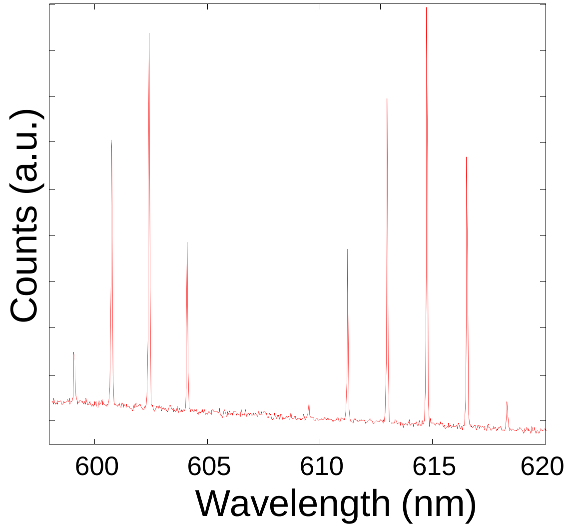
- Standard deviation below **5 µm**
- Acquisition time **14 µs**

Weimann et al., CLEO Conference, San Jose, USA, STh4O.3 (2014)

Polymer microgoblet lasers



$Q \approx 1.3 \cdot 10^7$ ($\lambda \approx 630$ nm)



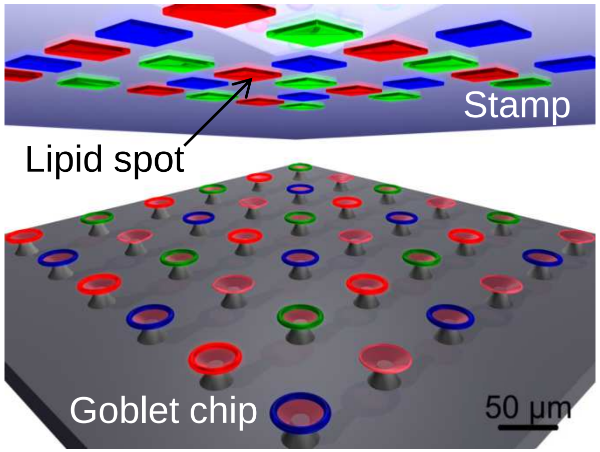
Sensing with Whispering-Gallery Mode (WGM) resonators:

- Coupling to functionalized resonator surface
- Detection by shift of resonance frequency
- High-Q resonators

⇒ Low detection limits, i.e., single virus, nanoparticle

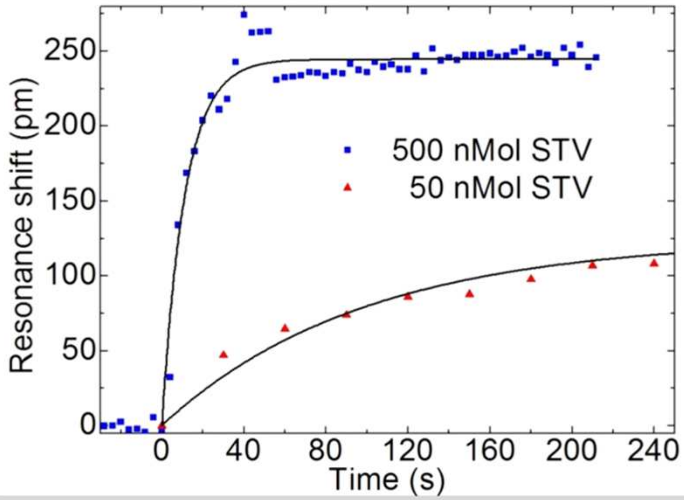
Grossmann et al., **Appl. Phys. Lett.** 96 (2010)

Selective label-free detection of biomolecules



U. Bog et al., **Small** in print (2014)

Parallel functionalization of whole micro-goblet chip using array of phospholipid spots on glass stamp



Fundamentals of Wave Propagation in Optics

Maxwell's equations and constitutive relations

Basic assumptions:

- No free charges, no currents (if needed, they are treated by a complex dielectric permittivity…)
- Nonmagnetic material
- Linear material (for now…)

Frequency-domain quantities: Either Fourier transforms or complex amplitudes of time-harmonic quantities

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = 0$$

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t}$$

$\circ\!\!-\!\!\bullet$

$$\nabla \cdot \underline{\mathbf{D}}(\mathbf{r},\omega) = 0$$

$$\nabla \times \underline{\mathbf{E}}(\mathbf{r},\omega) = -\mathrm{j}\,\omega \underline{\mathbf{B}}(\mathbf{r},\omega)$$

$$\nabla \cdot \underline{\mathbf{B}}(\mathbf{r},\omega) = 0$$

$$\nabla \times \underline{\mathbf{H}}(\mathbf{r},\omega) = \mathrm{j}\,\omega \underline{\mathbf{D}}(\mathbf{r},\omega)$$

$$\mathbf{B}(\mathbf{r},t) = \mu_0 \mathbf{H}(\mathbf{r},t)$$

$$\mathbf{D}(\mathbf{r},t) = \epsilon_0 \mathbf{E}(\mathbf{r},t) + \mathbf{P}(\mathbf{r},t)$$

$\circ\!\!-\!\!\bullet$

$$\underline{\mathbf{B}}(\mathbf{r},\omega) = \mu_0 \underline{\mathbf{H}}(\mathbf{r},\omega)$$

$$\underline{\mathbf{D}}(\mathbf{r},\omega) = \epsilon_0 \underline{\mathbf{E}}(\mathbf{r},\omega) + \underline{\mathbf{P}}(\mathbf{r},\omega)$$

Frequency-domain quantities and dielectric susceptibility

Fourier transformation:

$$\Psi(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{\Psi}(\omega)\, \mathrm{e}^{\mathrm{j}\omega t}\, \mathrm{d}\omega \quad \circ\!\!-\!\!\bullet \quad \tilde{\Psi}(\omega) = \int_{-\infty}^{+\infty} \Psi(t)\, \mathrm{e}^{-\mathrm{j}\omega t}\, \mathrm{d}t$$

Complex amplitude of a time-harmonic quantity:

$$\Psi(t) = \mathrm{Re}\left\{\underline{\Psi}(\omega) \exp\left(\mathrm{j}\,\omega t\right)\right\}$$

Note: For linear media materials, Maxwell's equations take the same form for Fourier transforms and complex time-domain amplitudes. In nonlinear optical media, the two quantities must be carefully distinguished!

Dielectric susceptibility of a linear medium:

Note: In optics, the term "linear medium" denotes a material, for which the polarization **P** depends linearly on the electric field **E**.

$$\mathbf{P}(\mathbf{r}, t) = \epsilon_0 \int_{-\infty}^{\infty} \chi(\mathbf{r}, \tau) \mathbf{E}(\mathbf{r}, t - \tau)\, \mathrm{d}\tau$$

Time domain: Convolution with a (causal) influence function

$$\circ\!\!-\!\!\bullet$$

$$\underline{\mathbf{P}}(\mathbf{r}, \omega) = \epsilon_0 \underline{\chi}(\mathbf{r}, \omega) \underline{\mathbf{E}}(\mathbf{r}, \omega)$$

Frequency domain: Multiplication with a complex transfer function

Constitutive relations and complex refractive index

Complex dielectric constant and refractive index :

$$\underline{\mathbf{D}}(\mathbf{r},\omega) = \epsilon_0 \underline{\mathbf{E}}(\mathbf{r},\omega) + \underline{\mathbf{P}}(\mathbf{r},\omega)$$
$$= \epsilon_0 \left(1 + \underline{\chi}(\mathbf{r},\omega)\right) \underline{\mathbf{E}}(\mathbf{r},\omega)$$
$$= \epsilon_0 \underline{\epsilon}_\mathrm{r}(\mathbf{r},\omega) \underline{\mathbf{E}}(\mathbf{r},\omega)$$
$$= \epsilon_0 \underline{n}^2(\mathbf{r},\omega) \underline{\mathbf{E}}(\mathbf{r},\omega).$$

Complex dielectric constant and refractive index :

$$\underline{\epsilon}_\mathrm{r}(\mathbf{r},\omega) = 1 + \underline{\chi}(\mathbf{r},\omega) = \underline{n}^2(\mathbf{r},\omega)$$

Convention: Positive values of n_i (ϵ_{ri}) are assigned to lossy media, negative values correspond to optical gain!

$$\underline{n} = n - \mathrm{j}\, n_i,$$
$$\epsilon_r = n^2 - n_i^2,$$
$$n^2 = \tfrac{1}{2}\epsilon_r \left(1 + \sqrt{1 + \epsilon_{ri}^2/\epsilon_r^2}\right),$$
$$n \approx \sqrt{\epsilon_r} \qquad (\text{for } |\epsilon_{ri}| \ll \epsilon_r)$$
$$n \approx \sqrt{|\epsilon_{ri}|/2} \qquad (\text{for } |\epsilon_{ri}| \gg \epsilon_r)$$

$$\underline{\epsilon}_r = \epsilon_r - \mathrm{j}\, \epsilon_{ri},$$
$$\epsilon_{ri} = 2 n n_i,$$
$$n_i = \epsilon_{ri}/(2n),$$
$$n_i \approx \epsilon_{ri}/(2\sqrt{\epsilon_r}),$$
$$n_i \approx \mathrm{sgn}(\epsilon_{ri})\sqrt{|\epsilon_{ri}|/2}.$$

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Basic relations of vector calculus

Linearität	
1. $\nabla(\alpha\Phi+\beta\Psi)=\alpha\,\nabla\Phi+\beta\,\nabla\Psi$	$\text{grad}(\alpha\Phi+\beta\Psi)=\alpha\,\text{grad}\,\Phi+\beta\,\text{grad}\,\Psi$
2. $\nabla\cdot(\alpha\mathbf{F}+\beta\mathbf{G})=\alpha\,\nabla\cdot\mathbf{F}+\beta\,\nabla\cdot\mathbf{G}$	$\text{div}(\alpha\mathbf{F}+\beta\mathbf{G})=\alpha\,\text{div}\,\mathbf{F}+\beta\,\text{div}\,\mathbf{G}$
3. $\nabla\times(\alpha\mathbf{F}+\beta\mathbf{G})=\alpha\,\nabla\times\mathbf{F}+\beta\,\nabla\times\mathbf{G}$	$\text{rot}(\alpha\mathbf{F}+\beta\mathbf{G})=\alpha\,\text{rot}\,\mathbf{F}+\beta\,\text{rot}\,\mathbf{G}$
Operation auf Produkten	
4. $\nabla(\Phi\Psi)=\Phi\,\nabla\Psi+\Psi\,\nabla\Phi$	$\text{grad}(\Phi\Psi)=\Phi\,\text{grad}\,\Psi+\Psi\,\text{grad}\,\Phi$
5. $\nabla(\mathbf{F}\cdot\mathbf{G})=(\mathbf{F}\cdot\nabla)\mathbf{G}+(\mathbf{G}\cdot\nabla)\mathbf{F}+$ $+\mathbf{F}\times(\nabla\times\mathbf{G})+\mathbf{G}\times(\nabla\times\mathbf{F})$	$\text{grad}(\mathbf{F}\cdot\mathbf{G})=(\mathbf{F}\cdot\text{grad})\mathbf{G}+$ $+(\mathbf{G}\cdot\text{grad})\mathbf{F}+\mathbf{F}\times\text{rot}\,\mathbf{G}+\mathbf{G}\times\text{rot}\,\mathbf{F}$
6. $\nabla\cdot(\Phi\mathbf{F})=\Phi\,\nabla\cdot\mathbf{F}+(\nabla\Phi)\cdot\mathbf{F}$	$\text{div}(\Phi\mathbf{F})=\Phi\,\text{div}\,\mathbf{F}+\mathbf{F}\cdot\text{grad}\,\Phi$
7. $\nabla\cdot(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot\nabla\times\mathbf{F}-\mathbf{F}\cdot\nabla\times\mathbf{G}$	$\text{div}(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot\text{rot}\,\mathbf{F}-\mathbf{F}\cdot\text{rot}\,\mathbf{G}$
8. $\nabla\times(\Phi\mathbf{F})=\Phi\,\nabla\times\mathbf{F}+(\nabla\Phi)\times\mathbf{F}$	$\text{rot}(\Phi\mathbf{F})=\Phi\,\text{rot}\,\mathbf{F}+(\text{grad}\,\Phi)\times\mathbf{F}$
9. $\nabla\times(\mathbf{F}\times\mathbf{G})=(\mathbf{G}\cdot\nabla)\mathbf{F}-(\mathbf{F}\cdot\nabla)\mathbf{G}+$ $+\mathbf{F}(\nabla\cdot\mathbf{G})-\mathbf{G}(\nabla\cdot\mathbf{F})$	$\text{rot}(\mathbf{F}\times\mathbf{G})=(\mathbf{G}\cdot\text{grad})\mathbf{F}-$ $-(\mathbf{F}\cdot\text{grad})\mathbf{G}+\mathbf{F}\,\text{div}\,\mathbf{G}-\mathbf{G}\,\text{div}\,\mathbf{F}$
Zweifache Anwendung von ∇	
10. $\nabla\cdot(\nabla\times\mathbf{F})=0$	$\text{div rot}\,\mathbf{F}=0$
11. $\nabla\times(\nabla\Phi)=\mathbf{0}$	$\text{rot grad}\,\Phi=\mathbf{0}$
12. $\nabla\times(\nabla\times\mathbf{F})=\nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F}$	$\text{rot rot}\,\mathbf{F}=\text{grad div}\,\mathbf{F}-\Delta\mathbf{F}$

Rade / Westergren, Mathematische Formeln, Springer

Wave equation and plane waves

General form:

$$\nabla^2 \underline{\mathbf{E}}(\mathbf{r},\omega) + \nabla\left(\frac{\nabla \underline{\epsilon}_\mathrm{r}(\mathbf{r},\omega)}{\underline{\epsilon}_\mathrm{r}(\mathbf{r},\omega)} \cdot \underline{\mathbf{E}}(\mathbf{r},\omega)\right) + k_0^2 \underline{\epsilon}_\mathrm{r}(\mathbf{r},\omega)\underline{\mathbf{E}}(\mathbf{r},\omega) = 0$$

$$\nabla^2 \underline{\mathbf{H}}(\mathbf{r},\omega) + \frac{\nabla \underline{\epsilon}_\mathrm{r}(\mathbf{r},\omega)}{\underline{\epsilon}_\mathrm{r}(\mathbf{r},\omega)} \times (\nabla \times \underline{\mathbf{H}}(\mathbf{r},\omega)) + k_0^2 \underline{\epsilon}_\mathrm{r}(\mathbf{r},\omega)\underline{\mathbf{H}}(\mathbf{r},\omega) = 0$$

where $k_0 = \frac{\omega}{c}$

Weakly inhomogeneous media: ϵ can be assumed constant within distances of the order of a wavelength

$$\nabla^2 \underline{\mathbf{E}}(\mathbf{r},\omega) + k_0^2 \underline{\epsilon}_\mathrm{r}(\mathbf{r},\omega)\underline{\mathbf{E}}(\mathbf{r},\omega) = 0$$

$$\nabla^2 \underline{\mathbf{H}}(\mathbf{r},\omega) + k_0^2 \underline{\epsilon}_\mathrm{r}(\mathbf{r},\omega)\underline{\mathbf{H}}(\mathbf{r},\omega) = 0$$

Solution for homogeneous media: Plane waves

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left\{\underline{\mathbf{E}}(\mathbf{r},\omega)\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\} = \mathrm{Re}\left\{\underline{\mathbf{E}}_0\,\mathrm{e}^{\mathrm{j}(\omega t - \underline{\mathbf{k}}\mathbf{r})}\right\}$$

$$\mathbf{H}(\mathbf{r},t) = \mathrm{Re}\left\{\underline{\mathbf{H}}(\mathbf{r},\omega)\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\} = \mathrm{Re}\left\{\underline{\mathbf{H}}_0\,\mathrm{e}^{\mathrm{j}(\omega t - \underline{\mathbf{k}}\mathbf{r})}\right\}$$

where $\underline{k}^2 = k_0^2 \underline{\epsilon}_\mathrm{r}(\omega)$

Properties of plane waves

- $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{H}_0$ are mutually connected and form an orthogonal right-handed system:

$$\mathbf{k} \cdot \underline{\mathbf{E}}_0 = 0 \qquad \underline{\mathbf{H}}_0 = \frac{1}{\omega \mu_0} \underline{\mathbf{k}} \times \underline{\mathbf{E}}_0$$

$$\mathbf{k} \cdot \underline{\mathbf{H}}_0 = 0 \qquad \underline{\mathbf{E}}_0 = -\frac{1}{\omega \epsilon_0 \epsilon_r} \underline{\mathbf{k}} \times \underline{\mathbf{H}}_0$$

- The attenuation of a plane wave is linked to the imaginary part n_i of the complex refractive index. For a plane wave propagating in positive z-direction, the power decays as $e^{-\alpha z}$, where the attenuation constant α is given by

$$\alpha = 2 k_0 n_i$$

Note: A positive value of n_i corresponds to a positive attenuation coefficient α and therefore to optical loss.

Mehrfache Produkte

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \qquad \textit{(Graßmann-Entwicklung)}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) \qquad \textit{(Lagrange-Identität)}$$

Rade / Westergren, Mathematische Formeln, Springer

Kramers-Kronig relations

Recall: The complex susceptibility is the Fourier transform of a causal influence function in time domain

$$\mathbf{P}(t) = \epsilon_0 \int_{-\infty}^{\infty} \chi(\tau)\mathbf{E}(t-\tau)\,\mathrm{d}\tau \quad \circ\!\!-\!\!\bullet \quad \underline{\mathbf{P}}(\omega) = \epsilon_0 \underline{\chi}(\omega)\underline{\mathbf{E}}(\omega)$$

$$\underline{\chi}(\omega) = \chi(\omega) + \mathrm{j}\,\chi_i(\omega).$$

As a consequence, the real and the imaginary part of the complex susceptibility are connected by the Hilbert transform,

"Cauchy principal value", i.e., the integration boundaries must approach the singularity "symmetrically" from both sides.

$$\chi(f) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi_i(f_0)}{f_0 - f}\,\mathrm{d}f_0$$

$$\chi_i(f) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi(f_0)}{f_0 - f}\,\mathrm{d}f_0$$

Making further use of the fact that $\chi(t)$ is real, the Kramers-Kronig relations can be derived,

$$\chi(f) = -\frac{2}{\pi}\mathcal{P}\int_{0}^{\infty} \frac{f_0\,\chi_i(f_0)}{f_0^2 - f^2}\,\mathrm{d}f_0$$

$$\chi_i(f) = \frac{2}{\pi}\mathcal{P}\int_{0}^{\infty} \frac{f\,\chi(f_0)}{f_0^2 - f^2}\,\mathrm{d}f_0$$

Kramers-Kronig relations - discussion

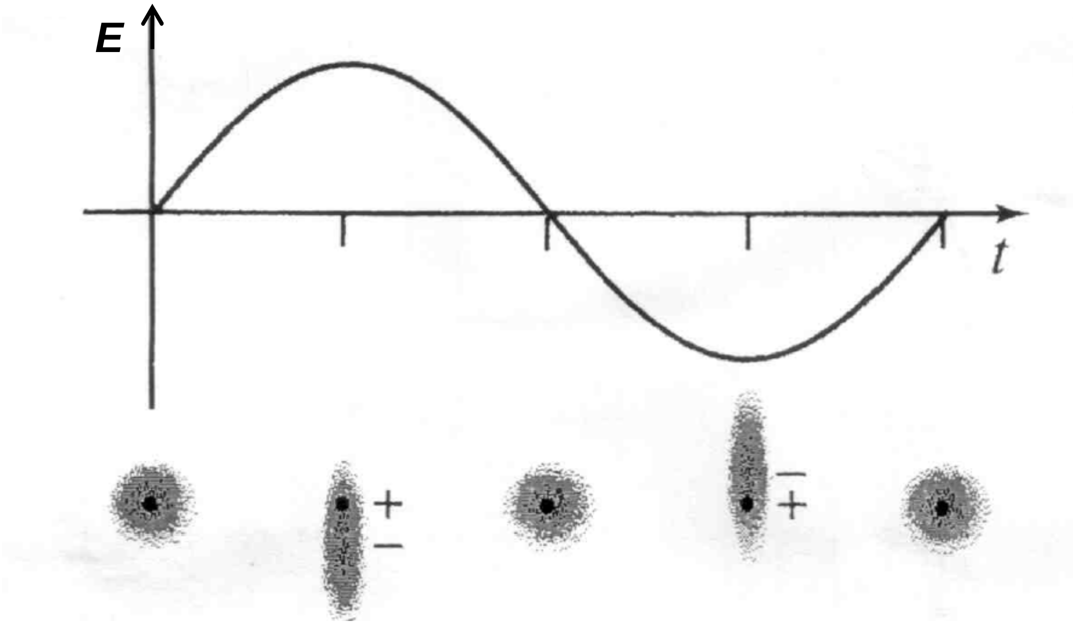
- The refractive index of a medium can be calculated from its absorption spectrum and vice versa. Absorption and dispersion are intimately related by fundamental principles.
- An "ideal" dispersionless lossless medium cannot exist:

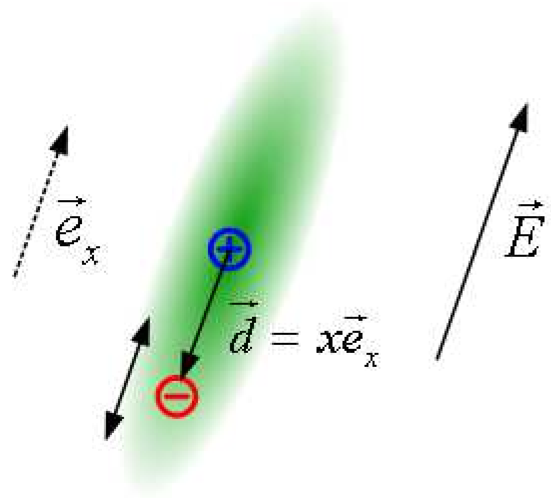
$$\chi_i(f) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi(f_0)}{f_0 - f}\,\mathrm{d}f_0 \qquad (1)$$

$$\chi(f) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi_i(f_0)}{f_0 - f}\,\mathrm{d}f_0 \qquad (2)$$

For constant $\chi(f)$, i.e., $\chi(f) = \epsilon_r(f) - 1 = \mathrm{const}_f$, we find $\chi_i(f) = \epsilon_{ri}(f) = 0$ from Eq. (1) , which implies $\chi(f) = 0$ and $\epsilon_r(f) = 1$, Eq. (2). Real media always have loss (or gain) in some frequency ranges, and the real part of the susceptibility is always frequency dependent. $\chi(f) = \mathrm{const}_f$ and $\chi_i = 0$ is only possible in certain frequency ranges.

Lorentz oscillator model of bound charges





Equation of motion for bound charges:

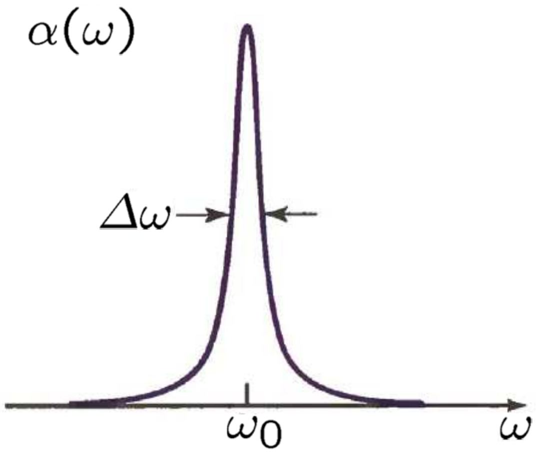
$$m_e \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -eE_x - m_e \omega_r^2 x - m_e \gamma_r \frac{\mathrm{d}x}{\mathrm{d}t}$$

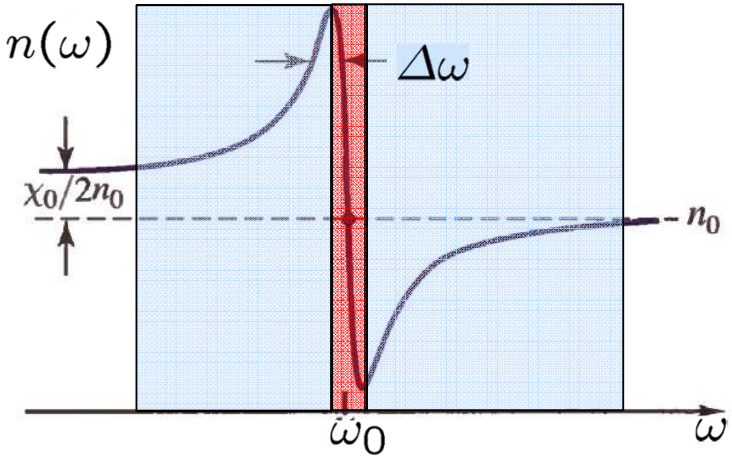
Complex electric susceptibility:

$$\underline{\chi}(\omega) = \chi_0 \frac{\omega_r^2}{\omega_r^2 - \omega^2 + j\omega\gamma_r}$$
$$= \frac{\left(\omega_r^2 - \omega^2\right)\omega_r^2}{\left(\omega_r^2 - \omega^2\right)^2 + \omega^2\gamma_r^2}\chi_0 - \mathrm{j}\,\frac{\omega\gamma_r\omega_r^2}{\left(\omega_r^2 - \omega^2\right)^2 + \omega^2\gamma_r^2}\chi_0.$$

Refractive index and absorption

Refractive index and absorption near a resonance:



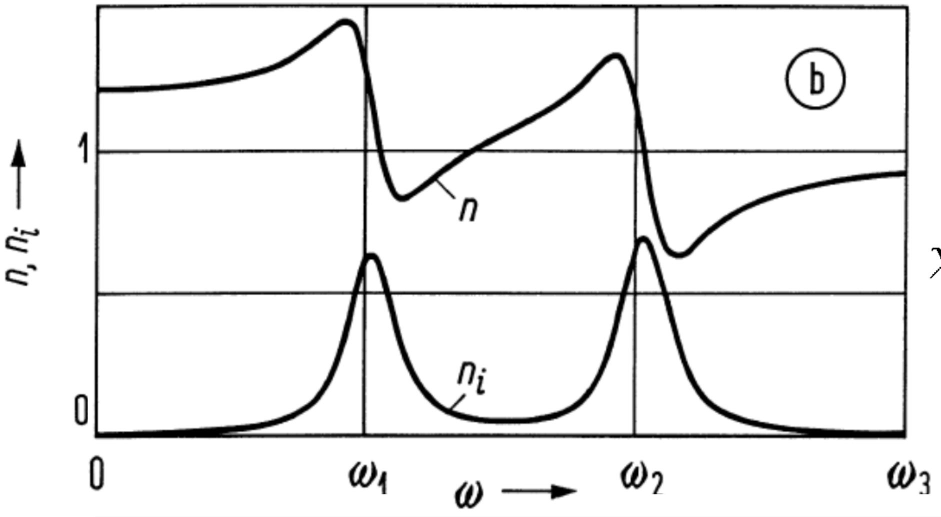


Adapted from Saleh, B. E. A. & Teich, M. C. (2007), *Fundamentals of Photonics, John Wiley & Sons, Hoboken, NJ.*

$$n \approx \sqrt{1+\chi}$$

$$n_i \approx -\frac{\chi_i}{2n}, \qquad \alpha = 2k_0 n_i$$

Real media often have several resonances, each of which contributes to the refractive index and to the absorption:



$$\chi(\omega) = \sum_\nu \frac{\left(\omega_{r\nu}^2 - \omega^2\right)\omega_r^2}{\left(\omega_{r\nu}^2 - \omega^2\right)^2 + \omega^2\gamma_{r\nu}^2}\chi_{0\nu}$$

$$\chi_i(\omega) = -\sum_\nu \frac{\omega\gamma_{r\nu}\omega_{r\nu}^2}{\left(\omega_{r\nu}^2 - \omega^2\right)^2 + \omega^2\gamma_{r\nu}^2}\chi_{0\nu}$$

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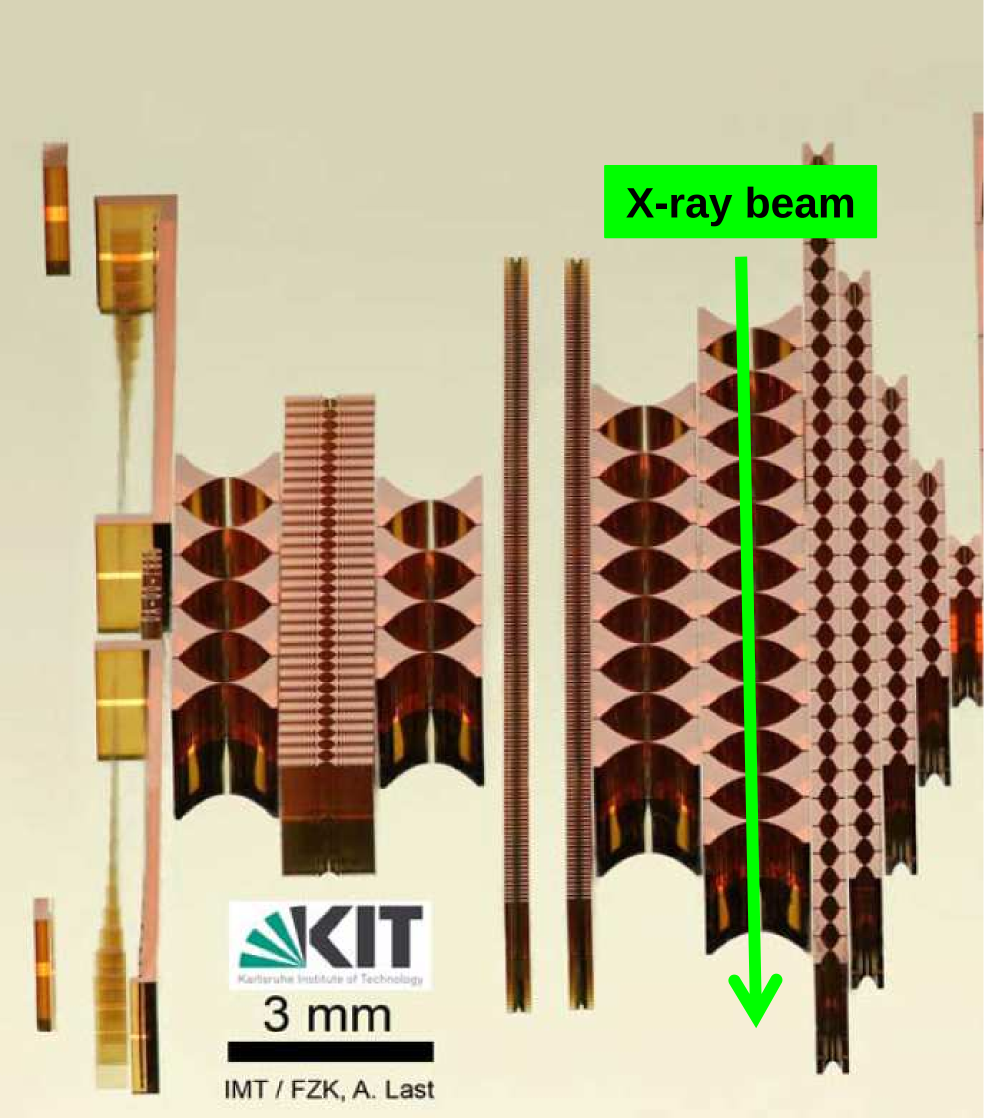
X-ray lenses



$$\chi(\omega) = \frac{\left(\omega_r^2 - \omega^2\right)\omega_r^2}{\left(\omega_r^2 - \omega^2\right)^2 + \omega^2\gamma_r^2}\chi_0$$

$$\chi_i(\omega) = -\frac{\omega\gamma_r\omega_r^2}{\left(\omega_r^2 - \omega^2\right)^2 + \omega^2\gamma_r^2}\chi_0$$

At very high frequencies ($\omega >> \omega_r$):

n < 1

$\Rightarrow$ Focussing lenses must have concave form!

n very close to 1 ($1 - n \approx 10^{-6}$)

$\Rightarrow$ Needs lots of lenses to obtain sufficient refraction.

Free charges

Equation of motion / complex susceptibility of bound charges:

$$m_e \frac{d^2x}{dt^2} = -eE_x - m_e\omega_r^2 x - m_e\gamma_r \frac{dx}{dt}, \qquad \underline{\chi}(\omega) = \frac{Ne^2}{\epsilon_0 m_e} \cdot \frac{1}{\omega_r^2 - \omega^2 + j\omega\gamma_r}$$

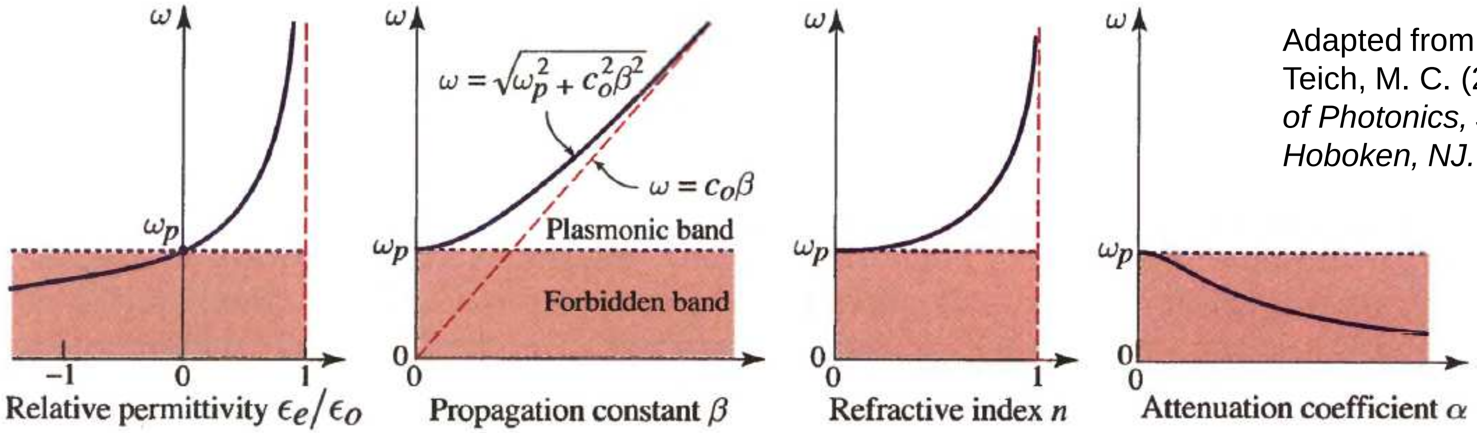
Free charges: Restoring force vanishes, i.e., $\omega_r = 0$:

$$\underline{\chi}(\omega) = -\frac{\omega_p^2}{\omega^2} \cdot \frac{1}{1 - \mathrm{j}\,\frac{\gamma_r}{\omega}} \approx -\frac{\omega_p^2}{\omega^2} \qquad \text{where} \qquad \omega_p^2 = \frac{Ne^2}{\epsilon_0 m_e}.$$

Discussion:

$\omega < \omega_p$: $\underline{n}$ and $\underline{k}$ are purely imaginary, i.e., the wave is attenuated and cannot propagate within the material ("forbidden band")

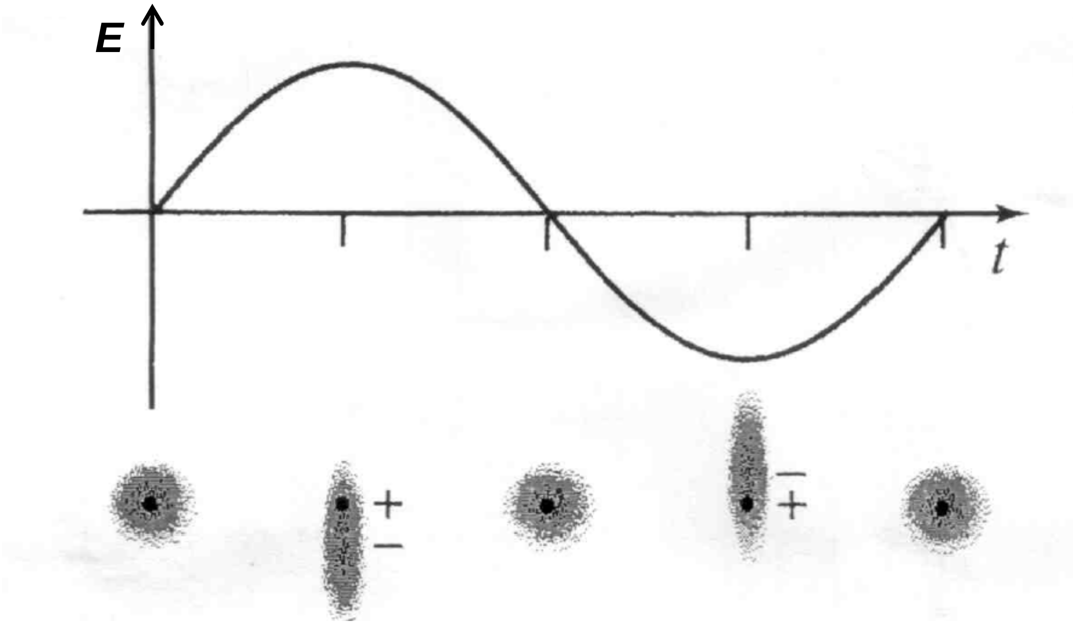
$\omega > \omega_p$: $\underline{n}$ and $\underline{k}$ are purely real, i.e., the metal behaves like a lossless dielectric with unique dispersion characteristics ("plasmonic band")

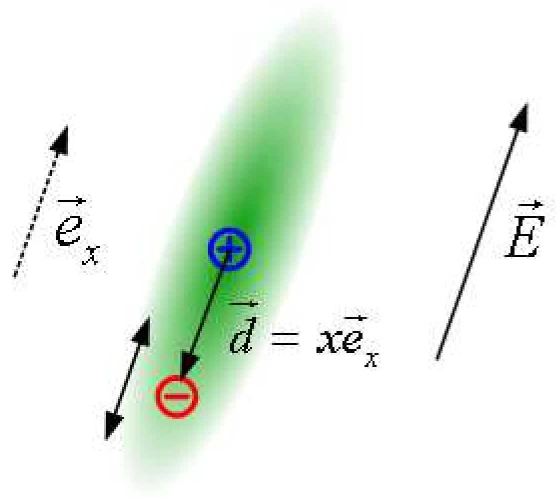


Adapted from Saleh, B. E. A. & Teich, M. C. (2007), *Fundamentals of Photonics, John Wiley & Sons, Hoboken, NJ.*

[Back to metal-clad slab waveguide…]

Lorentz oscillator model of bound charges





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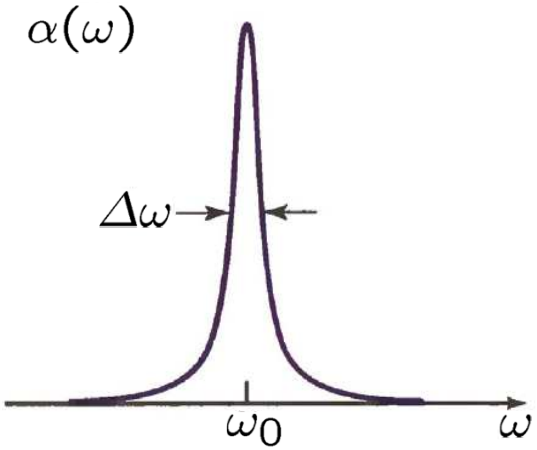
Complex electric susceptibility:

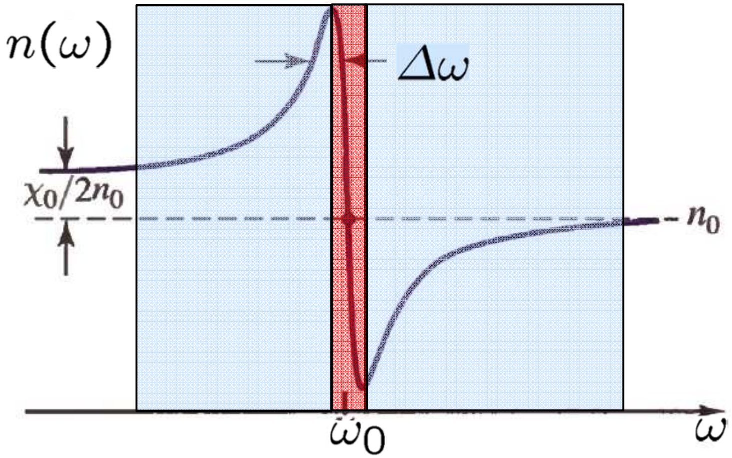
$$\underline{\chi}(\omega) = \chi_0 \frac{\omega_r^2}{\omega_r^2 - \omega^2 + j\omega\gamma_r}$$

$$= \frac{\left(\omega_r^2 - \omega^2\right)\omega_r^2}{\left(\omega_r^2 - \omega^2\right)^2 + \omega^2\gamma_r^2}\chi_0 - \mathrm{j}\,\frac{\omega\gamma_r\omega_r^2}{\left(\omega_r^2 - \omega^2\right)^2 + \omega^2\gamma_r^2}\chi_0 .$$

Refractive index and absorption

Refractive index and absorption near a resonance:





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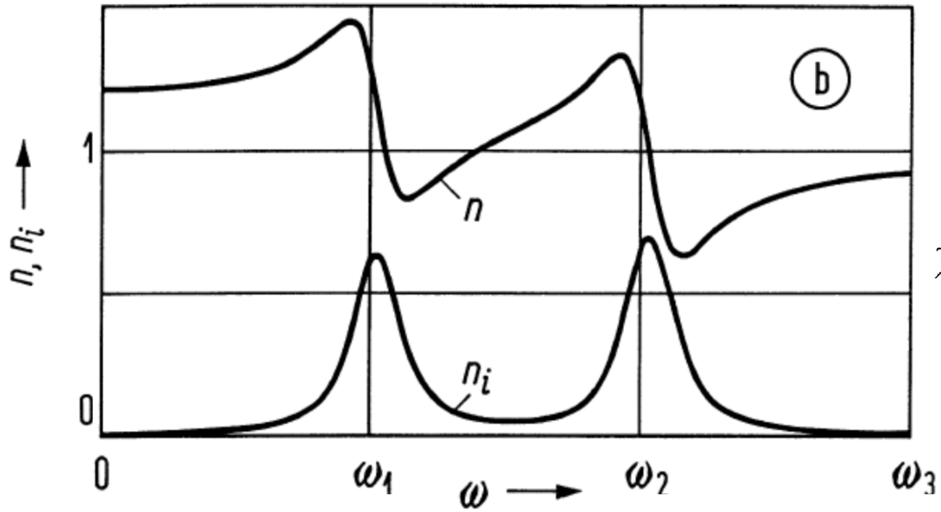
$$n \approx \sqrt{1+\chi}$$

$$n_i \approx -\frac{\chi_i}{2n}, \qquad \alpha = 2k_0 n_i$$

Real media often have several resonances, each of which contributes to the refractive index and to the absorption:



$$\chi(\omega) = \sum_{\nu} \frac{\left(\omega_{r\nu}^2 - \omega^2\right)\omega_r^2}{\left(\omega_{r\nu}^2 - \omega^2\right)^2 + \omega^2\gamma_{r\nu}^2}\chi_{0\nu}$$

$$\chi_i(\omega) = -\sum_{\nu} \frac{\omega\gamma_{r\nu}\omega_{r\nu}^2}{\left(\omega_{r\nu}^2 - \omega^2\right)^2 + \omega^2\gamma_{r\nu}^2}\chi_{0\nu}$$

Sellmeier equations

Complex electric susceptibility far from resonance ($|\omega_r - \omega| >> \gamma_r$):

$$\underline{\chi}(\omega) \approx \frac{\omega_r^2}{\omega_r^2 - \omega^2}\chi_0 \qquad \text{where} \qquad \chi_0 = \frac{Ne^2}{\epsilon_0 m_e \omega_r^2}$$

$\Rightarrow$ χ is approximately real, absorption is small. Contributions from multiple resonances lead to so-called Sellmeier equations:

$$n^2 = 1 + \chi = 1 + \sum_\nu \chi_{0\nu} \frac{f_\nu^2}{f_\nu^2 - f^2} = 1 + \sum_\nu \chi_{0\nu} \frac{\lambda^2}{\lambda^2 - \lambda_\nu^2}$$

For Sellmeier coefficients $\chi_{0\nu}$ and λ_ν, see reference books on optical materials or material databases, e.g.,

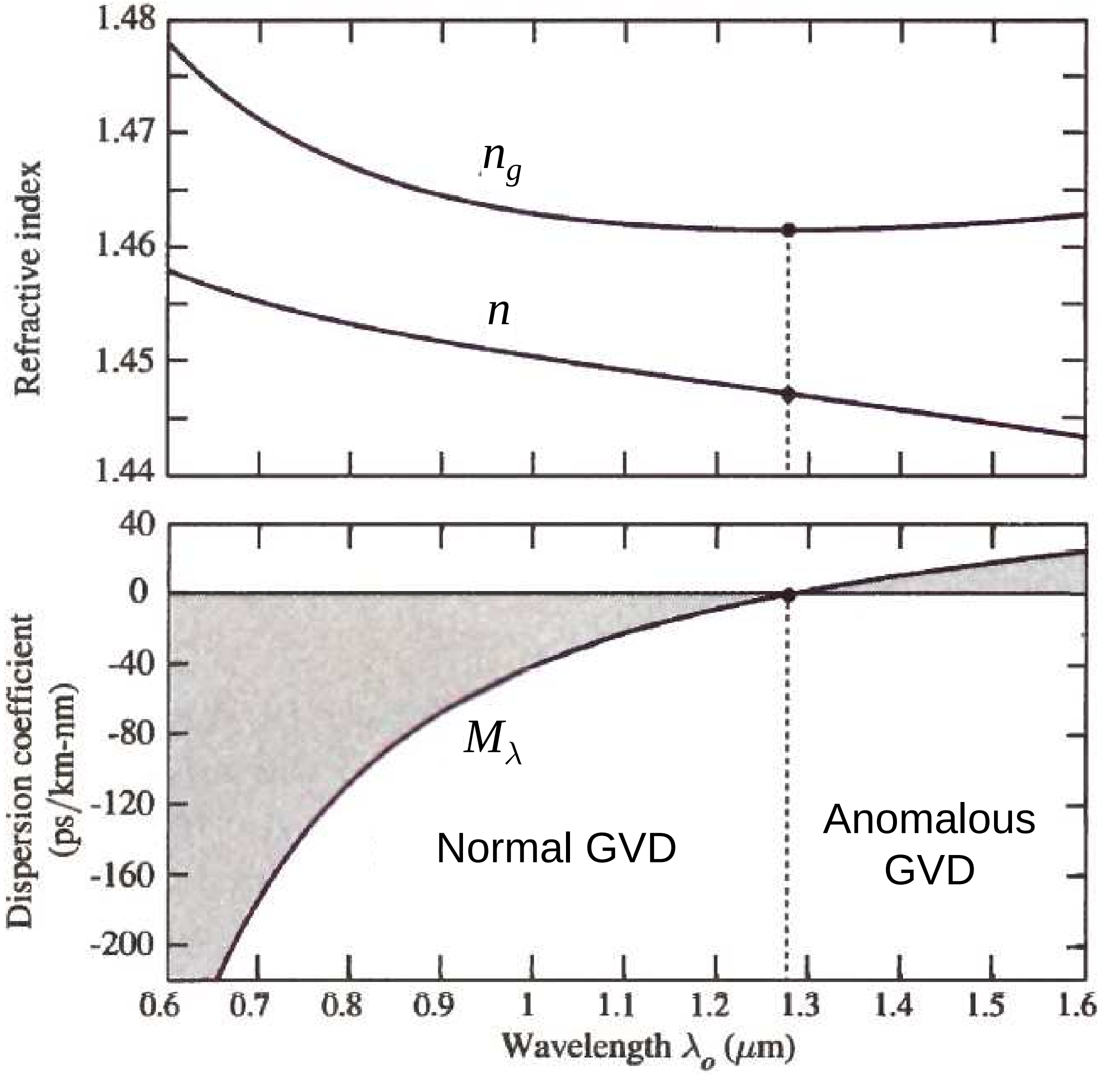
- Palik, E. D. (1998), *Handbook of Optical Constants of Solids, Academic Press, San Diego, CA*
- The Landolt Börnstein Database, *http://www.springermaterials.com/navigation/*

Sellmeier coefficients of various materials

Material	Sellmeier Equation (Wavelength λ in μm)	Wavelength Range (μm)
Fused silica	$n^2 = 1 + \frac{0.6962\lambda^2}{\lambda^2 - (0.06840)^2} + \frac{0.4079\lambda^2}{\lambda^2 - (0.1162)^2} + \frac{0.8975\lambda^2}{\lambda^2 - (9.8962)^2}$	0.21–3.71
Si	$n^2 = 1 + \frac{10.6684\lambda^2}{\lambda^2 - (0.3015)^2} + \frac{0.0030\lambda^2}{\lambda^2 - (1.1347)^2} + \frac{1.5413\lambda^2}{\lambda^2 - (1104.0)^2}$	1.36–11
GaAs	$n^2 = 3.5 + \frac{7.4969\lambda^2}{\lambda^2 - (0.4082)^2} + \frac{1.9347\lambda^2}{\lambda^2 - (37.17)^2}$	1.4–11
BBO	$n_o^2 = 2.7359 + \frac{0.01878}{\lambda^2 - 0.01822} - 0.01354\lambda^2$	0.22–1.06
	$n_e^2 = 2.3753 + \frac{0.01224}{\lambda^2 - 0.01667} - 0.01516\lambda^2$	
KDP	$n_o^2 = 1 + \frac{1.2566\lambda^2}{\lambda^2 - (0.09191)^2} + \frac{33.8991\lambda^2}{\lambda^2 - (33.3752)^2}$	0.4–1.06
	$n_e^2 = 1 + \frac{1.1311\lambda^2}{\lambda^2 - (0.09026)^2} + \frac{5.7568\lambda^2}{\lambda^2 - (28.4913)^2}$	
$LiNbO_3$	$n_o^2 = 2.3920 + \frac{2.5112\lambda^2}{\lambda^2 - (0.217)^2} + \frac{7.1333\lambda^2}{\lambda^2 - (16.502)^2}$	0.4–3.1
	$n_e^2 = 2.3247 + \frac{2.2565\lambda^2}{\lambda^2 - (0.210)^2} + \frac{14.503\lambda^2}{\lambda^2 - (25.915)^2}$	

Saleh, B. E. A. & Teich, M. C. (2007), *Fundamentals of Photonics, John Wiley & Sons, Hoboken, NJ.*

Wavelength-dependent refractive index for fused silica



Wavelength-dependent refractive index

⇒ Different spectral components of a time-dependent signal travel at different group velocities.

⇒ Deformation of signal shape due to dispersion.

Saleh, B. E. A. & Teich, M. C. (2007), *Fundamentals of Photonics, John Wiley & Sons, Hoboken, NJ.*

Pulse propagation in dispersive media

$$\underline{a}(0,t) = \underline{A}(0,t)\exp(\mathrm{j}\,\omega_c t)$$

Slowly varying envelope

$$\underline{\tilde{a}}(0,\omega) = \underline{\tilde{A}}(0,\omega-\omega_c)$$

Propagation: $\mathrm{e}^{-\mathrm{j}\,\beta(\omega)z}$

$$\underline{\tilde{a}}(z,\omega) = \underline{\tilde{A}}(0,\omega-\omega_c)\,\mathrm{e}^{-\mathrm{j}\,\beta(\omega)z}$$

$$\underline{a}(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underline{\tilde{A}}(0,\omega-\omega_c)\,\mathrm{e}^{-\mathrm{j}\,\beta(\omega)z}\,\mathrm{e}^{\mathrm{j}\,\omega t}\,\mathrm{d}\omega$$

Taylor expansion of propagation constant:

$$\beta(\omega) = \frac{\omega}{c}n(\omega) \approx \beta_c^{(0)} + (\omega-\omega_c)\beta_c^{(1)} + \frac{(\omega-\omega_c)^2}{2!}\beta_c^{(2)} + \frac{(\omega-\omega_c)^3}{3!}\beta_c^{(3)} + \dots$$

where $\beta_c^{(i)} = \left.\frac{\mathrm{d}^i\beta(\omega)}{\mathrm{d}\omega^i}\right|_{\omega=\omega_c}$

Retaining only the first two terms $\beta_c^{(0)}$ and $\beta_c^{(1)}$, the signal can be written as:

$$\underline{a}(z,t) = \underline{A}\left(0, t - \beta_c^{(1)} z\right) \mathrm{e}^{\mathrm{j}\left(\omega_c t - \beta_c^{(0)} z\right)} .$$

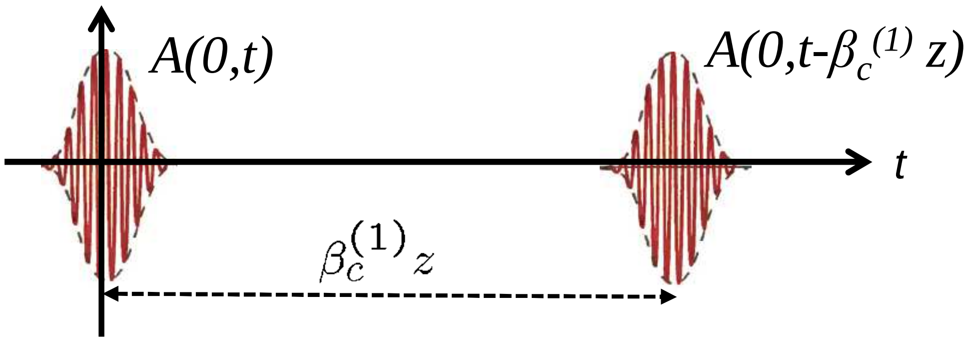
Phase shift of the carrier wave:

$$\beta_c^{(0)} z = \omega_c \frac{z}{v_p}, \qquad v_p = \frac{\omega_c}{\beta_c^{(0)}}, \qquad \beta_c^{(0)} = \frac{\omega_c}{c} n(\omega_c),$$

Group delay and group velocity of signal envelope:

$$t_g = \frac{z}{v_g} = \beta_c^{(1)} z ,$$

$$v_g = \frac{1}{\beta_c^{(1)}} = \frac{c}{n_g}.$$



Group refractive index:

$$n_g(\omega_c) = n(\omega_c) + \omega_c \left.\frac{\mathrm{d}n(\omega)}{\mathrm{d}\omega}\right|_{\omega=\omega_c} = n(\lambda_c) - \lambda_c \left.\frac{\mathrm{d}n(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_c} .$$

Group velocity dispersion (GVD)

Second- and higher-order terms in the Taylor series of $\beta(\omega)$ describe the frequency dependence of the group velocity.

Group delay spread of two packets centered at ω_c and $\omega_c + \Delta\omega_c$:

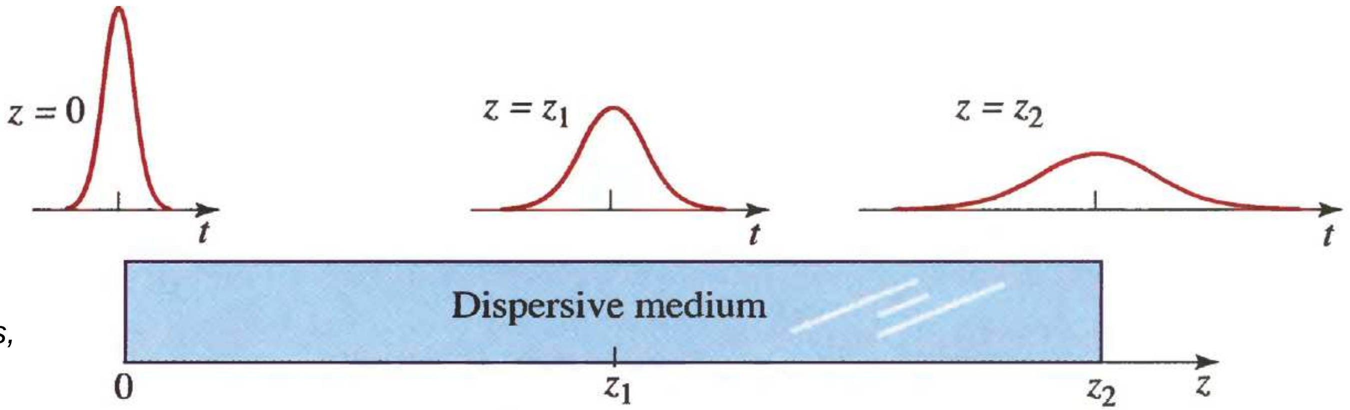
$$\triangle t_g = \left.\frac{\mathrm{d}t_g(\omega)}{\mathrm{d}\omega}\right|_{\omega=\omega_c} \times \Delta\omega_c = \beta_c^{(2)}\,\Delta\omega_c\,z$$

Material dispersion coefficients M_f and M_λ:

$$\frac{\triangle t_g}{z} = M_f\,\Delta f_c\,, \qquad M_f = \frac{2\pi}{c}\frac{\mathrm{d}n_g(\omega)}{\mathrm{d}\omega} = 2\pi\beta_c^{(2)}, \qquad \left[M_f\right] = \frac{\mathrm{s}}{\mathrm{m}\cdot\mathrm{Hz}}$$

$$\frac{\triangle t_g}{z} = M_\lambda\,\Delta\lambda_c\,, \qquad M_\lambda = \frac{1}{c}\frac{\mathrm{d}n_g(\lambda)}{\mathrm{d}\lambda} = -\frac{c}{\lambda^2}M_f = -\frac{2\pi c}{\lambda^2}\beta_c^{(2)}, \qquad [M_\lambda] = \frac{\mathrm{ps}}{\mathrm{km}\cdot\mathrm{nm}}$$

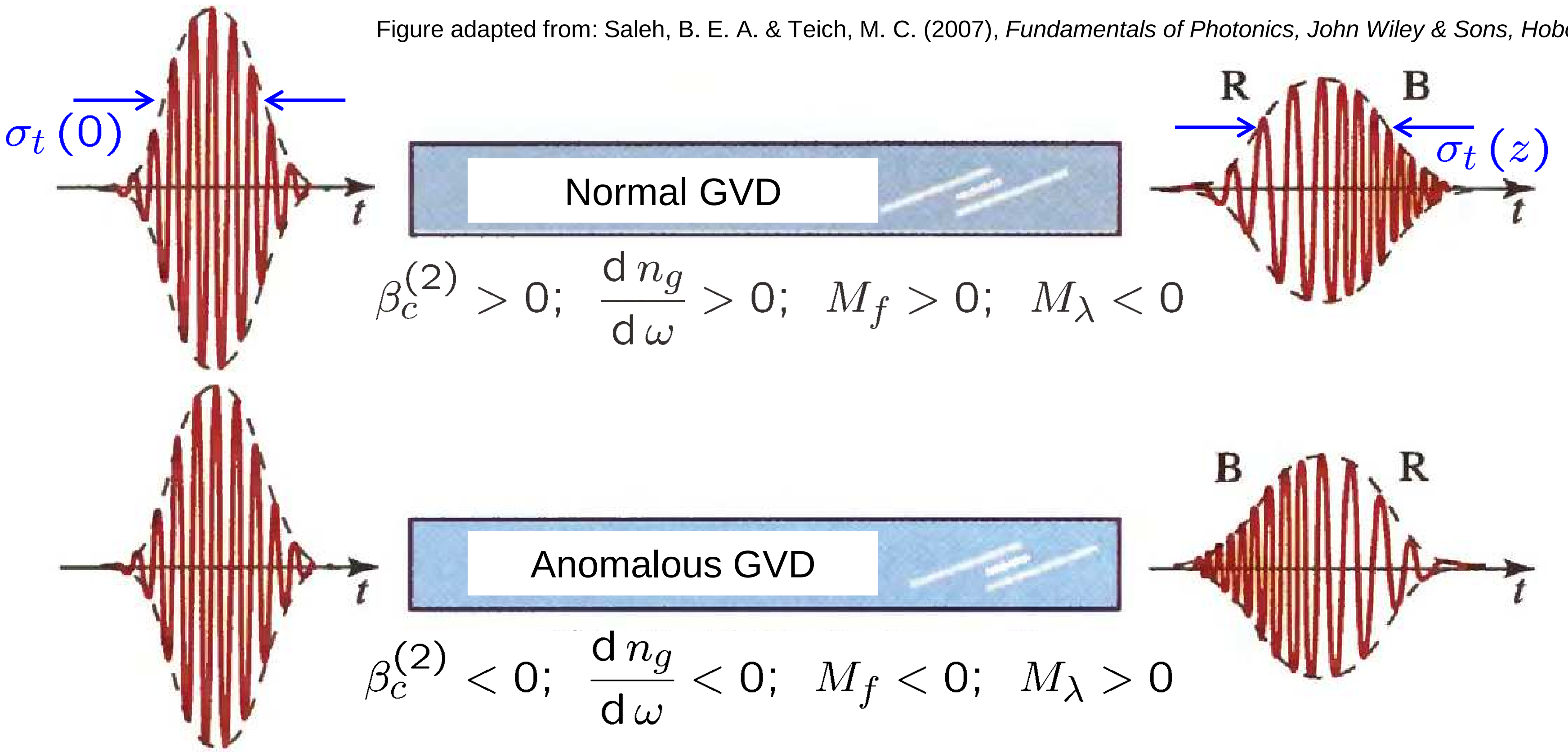
Example: Dispersive broadening of a Gaussian impulse



Saleh, B. E. A. & Teich, M. C. (2007), *Fundamentals of Photonics, John Wiley & Sons, Hoboken, NJ.*

Dispersive broadening of a Gaussian Impulse

Figure adapted from: Saleh, B. E. A. & Teich, M. C. (2007), *Fundamentals of Photonics, John Wiley & Sons, Hoboken, NJ.*



$$\beta_c^{(2)} > 0; \quad \frac{\mathrm{d}\, n_g}{\mathrm{d}\,\omega} > 0; \quad M_f > 0; \quad M_\lambda < 0$$

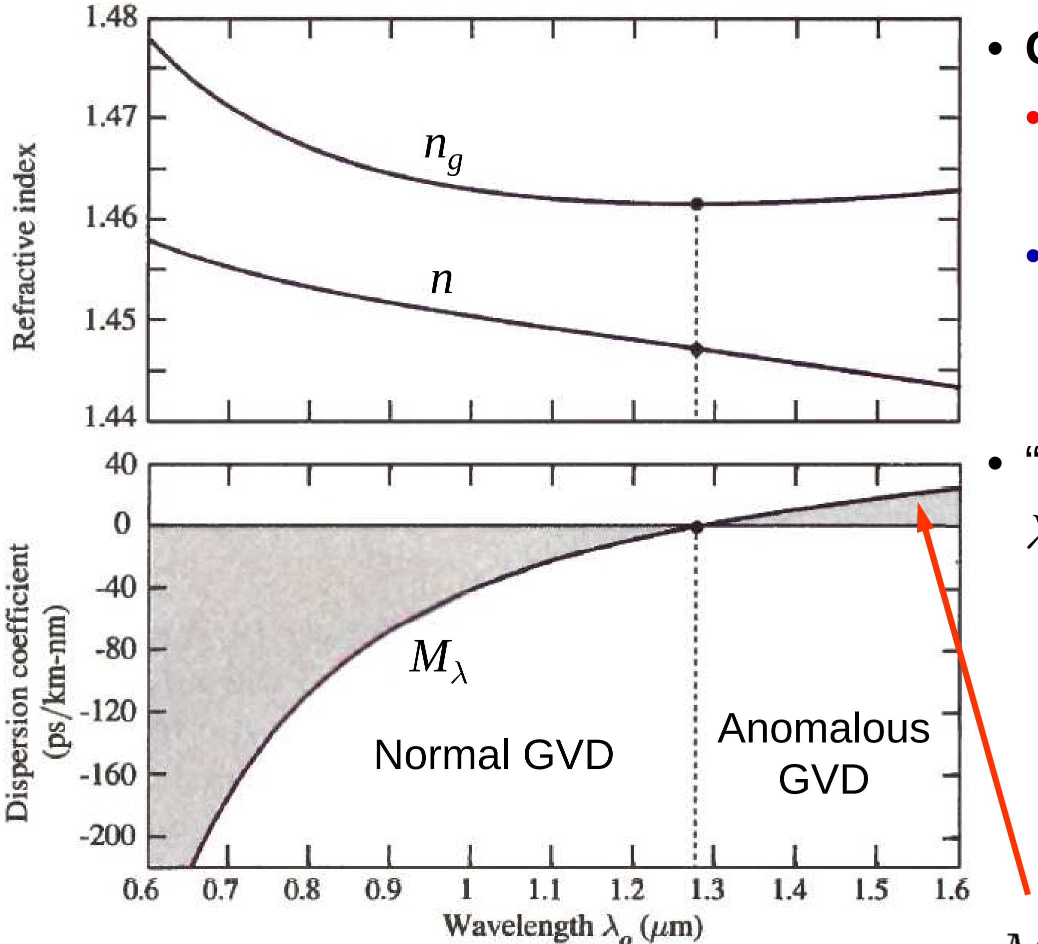
$$\beta_c^{(2)} < 0; \quad \frac{\mathrm{d}\, n_g}{\mathrm{d}\,\omega} < 0; \quad M_f < 0; \quad M_\lambda > 0$$

Problem set: Quantitative analysis

Gaussian impulse: $\underline{a}(0,t) = \underline{A}_o \exp\left(-\frac{t^2}{2\sigma_t^2}\right) \exp(\mathrm{j}\,\omega_c t)$

Dispersive broadening along z: $\sigma_t(z) = \sqrt{\sigma_t^2(0) + \frac{\left(\beta_c^{(2)} z\right)^2}{\sigma_t^2(0)}}$

Dispersive properties of fused silica



- **Group velocity dispersion (GVD)**:
  - Normal GVD below ~ 1.3 µm:
    $dn_g/d\omega > 0$, $dn_g/d\lambda < 0$, $M_\lambda < 0$
  - Anomalous GVD below ~ 1.3 µm:
    $dn_g/d\omega < 0$, $dn_g/d\lambda > 0$, $M_\lambda > 0$

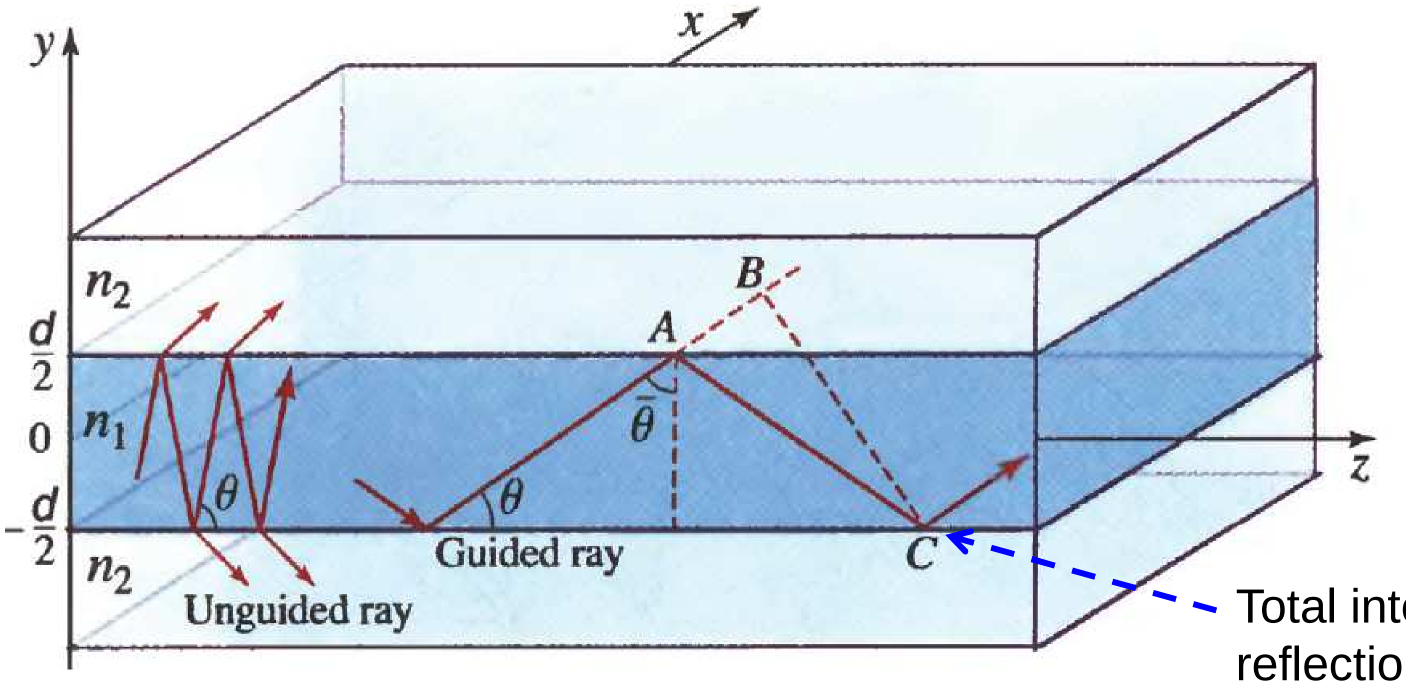
- “**Zero material dispersion wavelength**“
  $\lambda \approx$ 1.3 µm: Really zero dispersion?

$$M_\lambda(1.55\,\mu\text{m}) = \frac{22\,\text{ps}}{\text{km nm}}$$

Figure adapted from: Saleh, B. E. A. & Teich, M. C. (2007), *Fundamentals of Photonics, John Wiley & Sons, Hoboken, NJ.*

Optical Slab Waveguides

Dielectric slab waveguides



Ray-optics picture of a dieletric slab waveguide cannot explain a number of import effects, e.g. the formation of waveguide modes

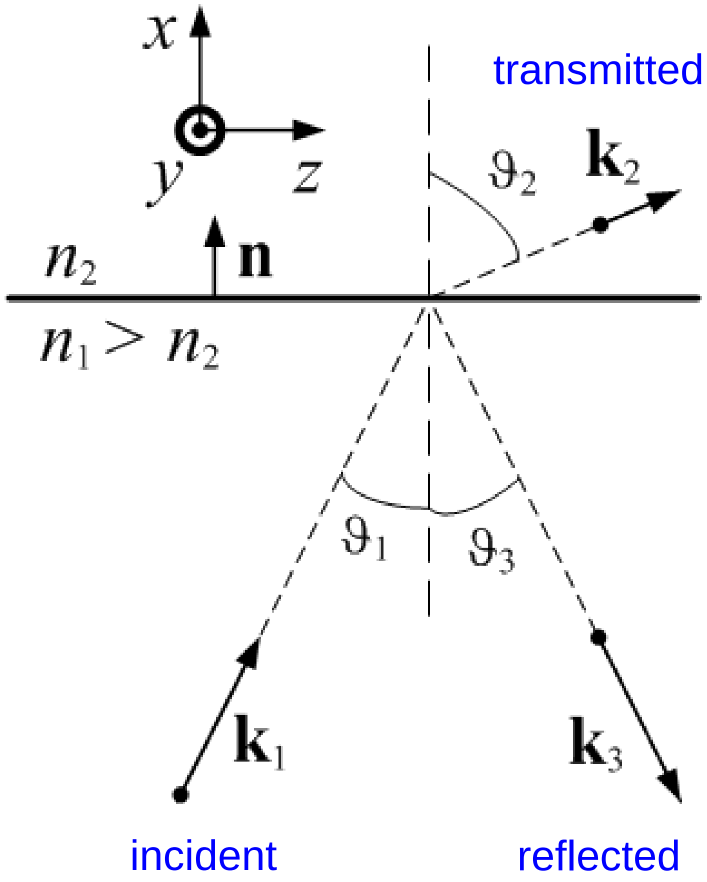
⇒ Electromagnetic model needed

Procedure:

- Reflection from plane dielectric boundary
- Lateral self-consistence and formation of modes
- Waveguide dispersion
- Extension to 3D geometries

Figure adapted from: Saleh, B. E. A. & Teich, M. C. (2007), *Fundamentals of Photonics, John Wiley & Sons, Hoboken, NJ.*

Plane dielectric boundary: Plane-wave ansatz



- Incident wave:

$$\underline{\mathbf{E}}_\mathrm{i} = \underline{\mathbf{E}}_1 \exp\left(-\mathrm{j}\,\mathbf{k}_1 \cdot \mathbf{r}\right), \qquad \mathbf{k}_1 = n_1 k_0 \begin{pmatrix} \cos(\vartheta_1) \\ 0 \\ \sin(\vartheta_1) \end{pmatrix}$$

$$\underline{\mathbf{H}}_\mathrm{i} = \frac{1}{\omega\mu_0}\mathbf{k}_1 \times \underline{\mathbf{E}}_1 \exp\left(-\mathrm{j}\,\mathbf{k}_1 \cdot \mathbf{r}\right)$$

- Transmitted wave:

$$\underline{\mathbf{E}}_\mathrm{t} = \underline{\mathbf{E}}_2 \exp\left(-\mathrm{j}\,\mathbf{k}_2 \cdot \mathbf{r}\right), \qquad \mathbf{k}_2 = n_2 k_0 \begin{pmatrix} \cos(\vartheta_2) \\ 0 \\ \sin(\vartheta_2) \end{pmatrix}$$

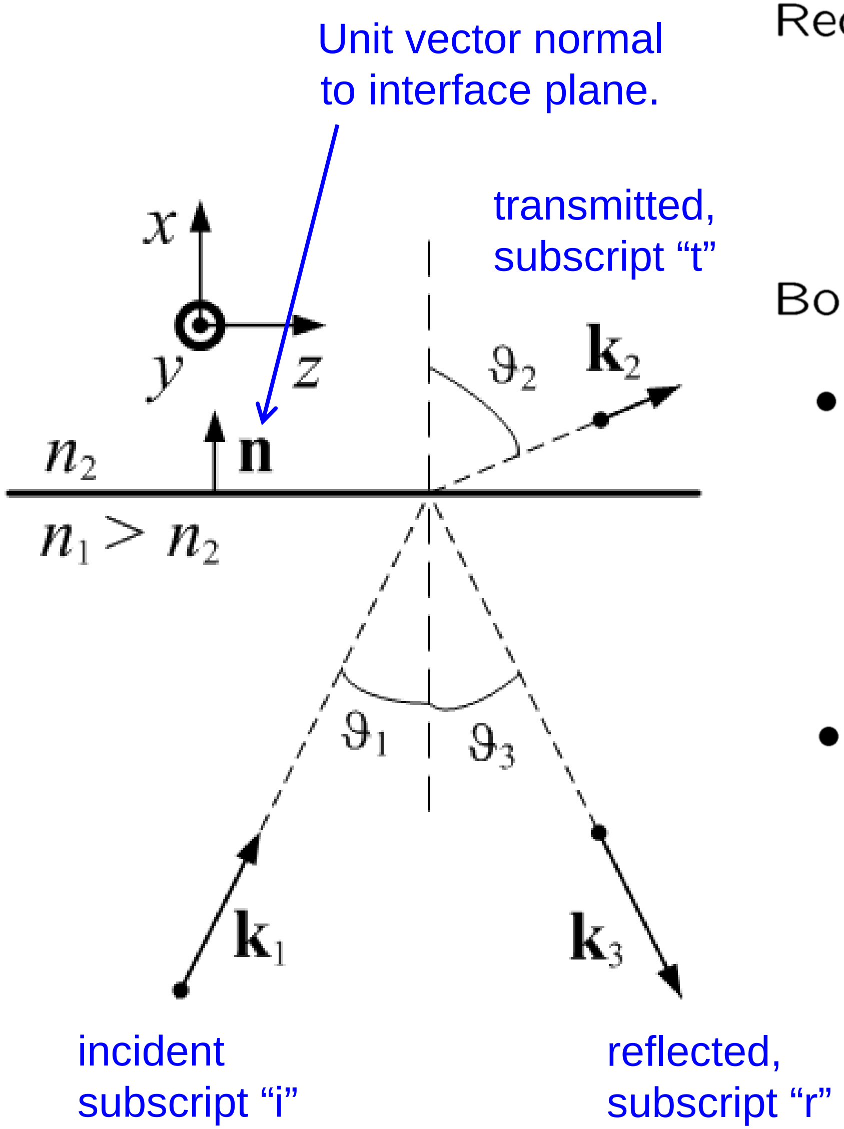
$$\underline{\mathbf{H}}_\mathrm{t} = \frac{1}{\omega\mu_0}\mathbf{k}_2 \times \underline{\mathbf{E}}_2 \exp\left(-\mathrm{j}\,\mathbf{k}_2 \cdot \mathbf{r}\right)$$

- Reflected wave:

$$\underline{\mathbf{E}}_\mathrm{r} = \underline{\mathbf{E}}_3 \exp\left(-\mathrm{j}\,\mathbf{k}_3 \cdot \mathbf{r}\right), \qquad \mathbf{k}_3 = n_1 k_0 \begin{pmatrix} -\cos(\vartheta_3) \\ 0 \\ \sin(\vartheta_3) \end{pmatrix}$$

$$\underline{\mathbf{H}}_\mathrm{r} = \frac{1}{\omega\mu_0}\mathbf{k}_3 \times \underline{\mathbf{E}}_3 \exp\left(-\mathrm{j}\,\mathbf{k}_3 \cdot \mathbf{r}\right)$$

Boundary conditions for *E*- and *H*-field



Recall:

$$\underline{\mathbf{D}} = \epsilon_0 \underline{n}^2 \underline{\mathbf{E}}$$
$$\underline{\mathbf{B}} = \mu_0 \underline{\mathbf{H}}$$

Boundary conditions at $x = 0$:

- Normal components of $\mathbf{D}$ and $\mathbf{B}$ are continuous:

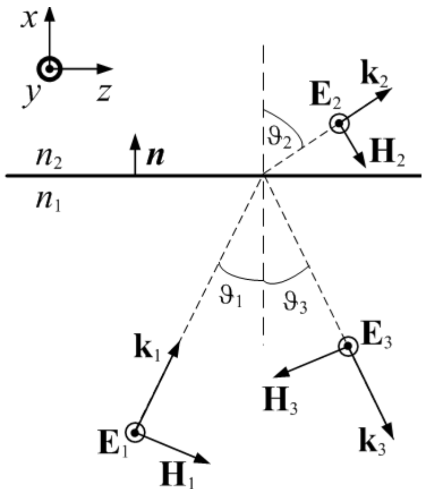
$$\left(n_1^2 \underline{\mathbf{E}}_\mathrm{i} + n_1^2 \underline{\mathbf{E}}_\mathrm{r} - n_2^2 \underline{\mathbf{E}}_\mathrm{t}\right) \cdot \mathbf{n} = 0$$
$$(\mathbf{k}_1 \times \underline{\mathbf{E}}_\mathrm{i} + \mathbf{k}_3 \times \underline{\mathbf{E}}_\mathrm{r} - \mathbf{k}_2 \times \underline{\mathbf{E}}_\mathrm{t}) \cdot \mathbf{n} = 0$$

- Tangential components of $\mathbf{E}$ and $\mathbf{H}$ are continuous:

$$(\underline{\mathbf{E}}_\mathrm{i} + \underline{\mathbf{E}}_\mathrm{r} - \underline{\mathbf{E}}_\mathrm{t}) \times \mathbf{n} = 0$$
$$(\mathbf{k}_1 \times \underline{\mathbf{E}}_\mathrm{i} + \mathbf{k}_3 \times \underline{\mathbf{E}}_\mathrm{r} - \mathbf{k}_2 \times \underline{\mathbf{E}}_\mathrm{t}) \times \mathbf{n} = 0$$

TE- and TM-polarization

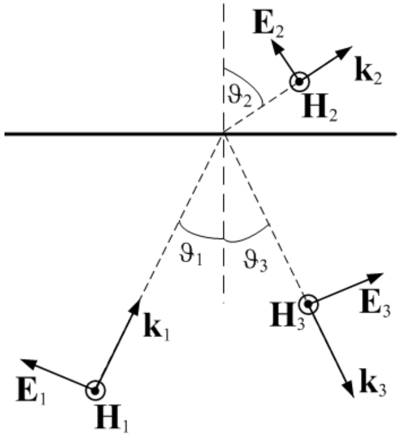
Strategy: Consider boundary conditions separately for two orthogonal linear polarizations! Any other polarization can be interpreted as a superposition of the two cases.



Transverse-electric wave (TE):

$\underline{\mathbf{E}}_\mathsf{i} \perp$ plane of incidence; $\underline{\mathbf{H}}_\mathsf{i} \parallel$ plane of incidence: TE-wave (transversal E-component) or H-wave (longitudinal H-component),

$$\underline{\mathbf{E}}_\mathsf{i} = \begin{pmatrix} 0 \\ E_{\mathsf{i}y} \\ 0 \end{pmatrix}, \qquad \underline{\mathbf{H}}_\mathsf{i} = \begin{pmatrix} H_{\mathsf{i}x} \\ 0 \\ H_{\mathsf{i}z} \end{pmatrix}.$$

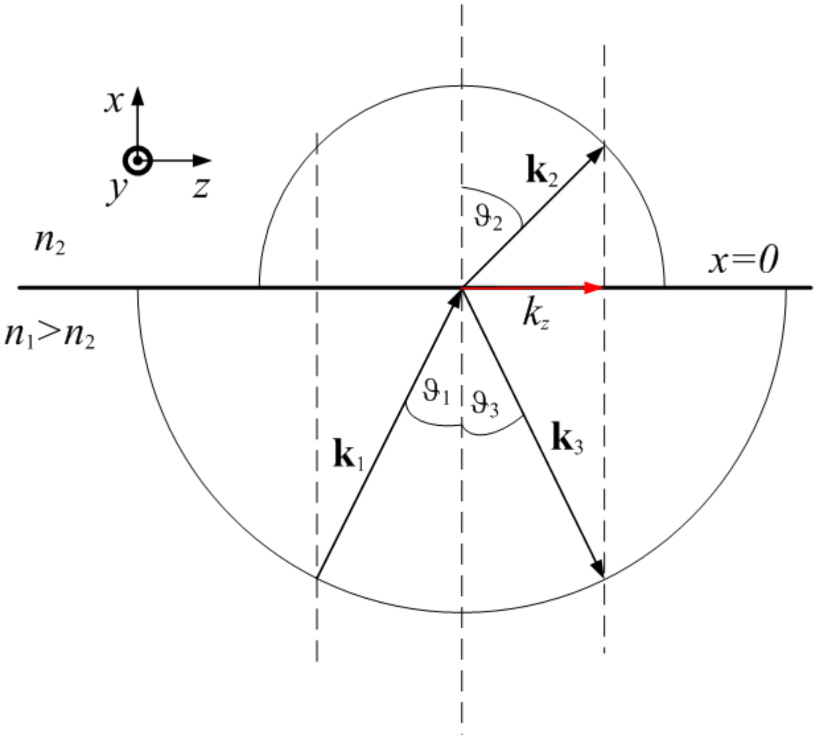


Transverse-magnetic wave (TM):

$\underline{\mathbf{H}}_\mathsf{i} \perp$ plane of incidence; $\underline{\mathbf{E}}_\mathsf{i} \parallel$ plane of incidence: TM-wave (transversal H-component) or E-wave (longitudinal E-component),

$$\underline{\mathbf{H}}_\mathsf{i} = \begin{pmatrix} 0 \\ H_{\mathsf{i}y} \\ 0 \end{pmatrix}, \qquad \underline{\mathbf{E}}_\mathsf{i} = \begin{pmatrix} E_{\mathsf{i}x} \\ 0 \\ E_{\mathsf{i}z} \end{pmatrix}.$$

Snell's law and law of reflection



Boundary conditions must be satisfied in all points of the (y, z)-plane simultaneously. Independent of the nature of the boundary conditions, the spatial variation of the fields must hence be the same,

$$\mathbf{k}_1 \cdot \mathbf{r} = \mathbf{k}_2 \cdot \mathbf{r} = \mathbf{k}_3 \cdot \mathbf{r} \qquad \text{for} \qquad \mathbf{r} = (0, y, z)^{\mathsf{T}}$$

Law of reflection:

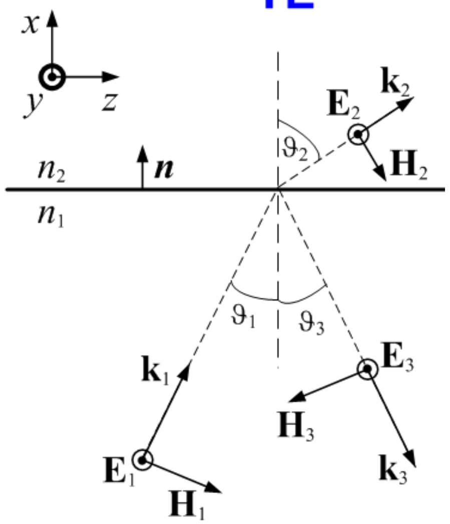
$$\vartheta_1 = \vartheta_3$$

Snell's law:

$$n_1 \sin(\vartheta_1) = n_2 \sin(\vartheta_2)$$

Reflection and transmission coefficients

TE

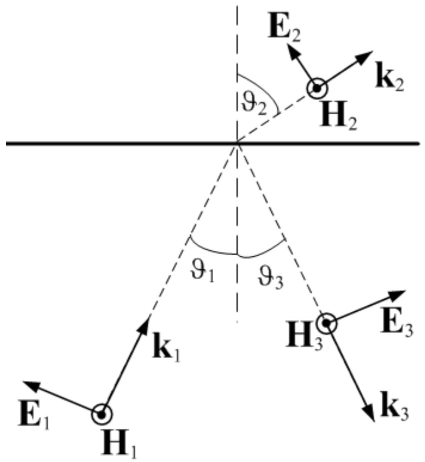


Amplitude reflection and transmission factors:

$$R_{\mathrm{TE}} = \frac{E_{3y}}{E_{1y}} = \frac{k_{1x} - k_{2x}}{k_{1x} + k_{2x}} = \frac{n_1 \cos\vartheta_1 - n_2 \cos\vartheta_2}{n_1 \cos\vartheta_1 + n_2 \cos\vartheta_2}$$

$$T_{\mathrm{TE}} = \frac{E_{2y}}{E_{1y}} = 1 + R_{\mathrm{TE}} = \frac{2k_{1x}}{k_{1x} + k_{2x}} = \frac{2n_1 \cos\vartheta_1}{n_1 \cos\vartheta_1 + n_2 \cos\vartheta_2}$$

TM

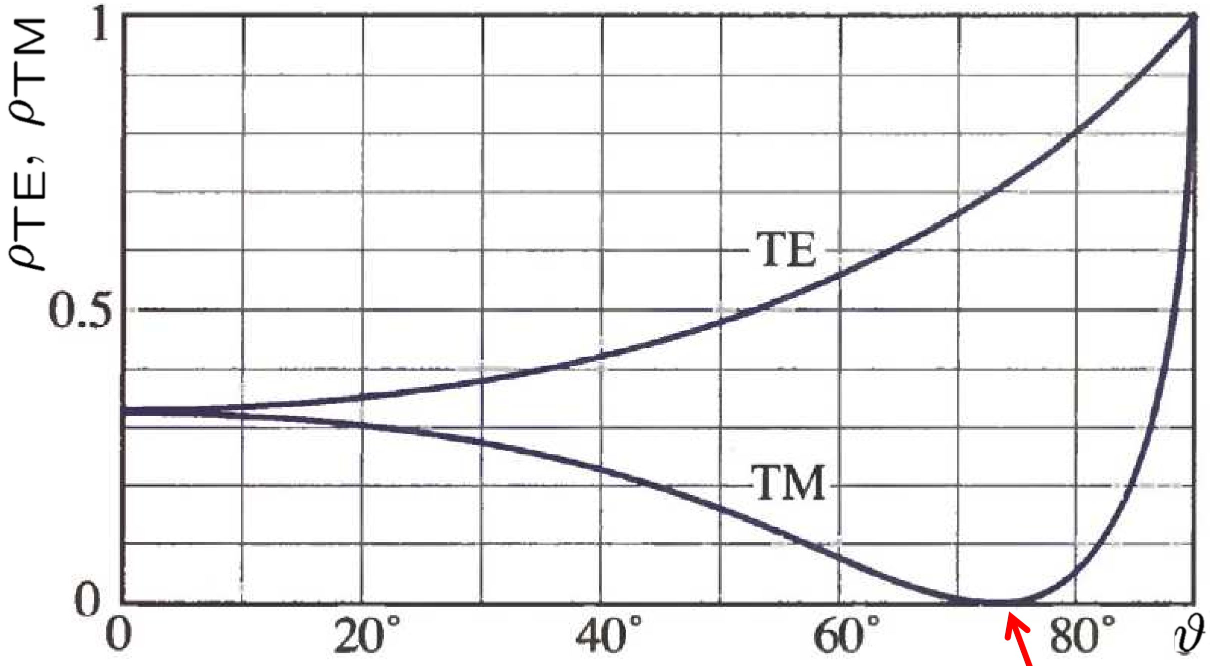


Amplitude reflection and transmission factors:

$$R_{\mathrm{TM}} = \frac{H_{3y}}{H_{1y}} = \frac{n_2^2 k_{1x} - n_1^2 k_{2x}}{n_2^2 k_{1x} + n_1^2 k_{2x}} = \frac{n_2 \cos\vartheta_1 - n_1 \cos\vartheta_2}{n_2 \cos\vartheta_1 + n_1 \cos\vartheta_2}$$

$$T_{\mathrm{TM}} = \frac{H_{2y}}{H_{1y}} = 1 + R_{\mathrm{TM}} = \frac{2n_2^2 k_{1x}}{n_2^2 k_{1x} + n_1^2 k_{2x}} = \frac{2n_2 \cos\vartheta_1}{n_2 \cos\vartheta_1 + n_1 \cos\vartheta_2}$$

Power transmission and reflection



Plane boundary between air (n_1 = 1) and GaAs (n_2 = 3.6)

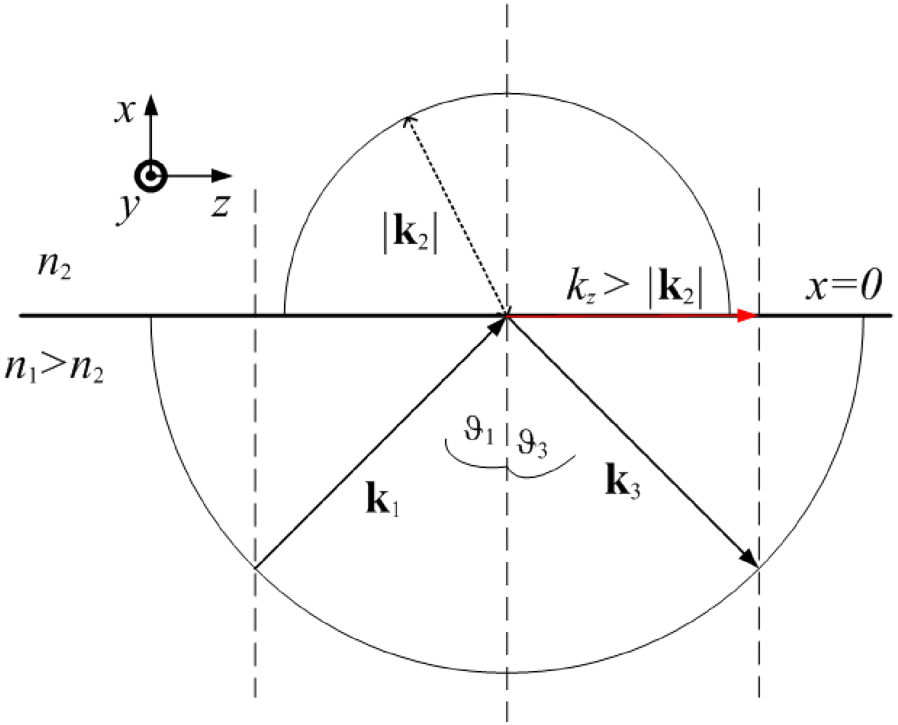
Power reflection:

$$\rho_{TE} = |R_{TE}|^2$$
$$\rho_{TM} = |R_{TM}|^2$$

Brewster angle: $\tan \vartheta_{1B} = \frac{n_2}{n_1}$

Saleh, B. E. A. & Teich, M. C. (2007), *Fundamentals of Photonics, John Wiley & Sons, Hoboken, NJ.*

Total internal reflection (TIR)



Limiting angle for total internal reflection:

$$\sin \vartheta_{1\mathrm{T}} = \frac{n_2}{n_1}$$

For $\vartheta_1 > \vartheta_{1\mathrm{T}}$:

$$k_{2z} = k_{1z} = n_1 k_0 \sin \vartheta_1 > n_2 k_0$$

$$k_{2x}^2 = n_2^2 k_0^2 - k_{2z}^2 < 0$$

$$k_{2x} = \pm \mathrm{j}\, k_0 \sqrt{n_1^2 \sin^2 \vartheta_1 - n_2^2} = \pm \mathrm{j}\, k_{2x}^{(\mathrm{i})}$$

$$\underline{\mathbf{E}}_\mathrm{t} = \underline{\mathbf{E}}_2 \exp\left(-\mathrm{j}\, \mathbf{k}_2 \cdot \mathbf{r}\right) = \underline{\mathbf{E}}_2 \exp\left(-\mathrm{j}\, k_{2z} z\right) \exp\left(-k_{2x}^{(\mathrm{i})} x\right),$$

i.e., $\underline{\mathbf{E}}_\mathrm{t}$ is evanescent in *x*-direction

Reflection factors for TIR

TE:

$$R_{\mathrm{TE}} = \frac{E_{3y}}{E_{1y}} = \frac{k_{1x} + \mathrm{j}\, k_{2x}^{(\mathrm{i})}}{k_{1x} - \mathrm{j}\, k_{2x}^{(\mathrm{i})}} = \exp\left(\mathrm{j}\, \varphi_{\mathrm{TE}}\right)$$

$$\varphi_{\mathrm{TE}} = 2 \arctan\left(\frac{k_{2x}^{(\mathrm{i})}}{k_{1x}}\right) = 2 \arctan\left(\frac{\sqrt{n_1^2 \sin^2 \vartheta_1 - n_2^2}}{n_1 \cos \vartheta_1}\right)$$

TM:

$$R_{\mathrm{TM}} = \frac{H_{3y}}{H_{1y}} = \frac{n_2^2 k_{1x} + \mathrm{j}\, n_1^2 k_{2x}^{(\mathrm{i})}}{n_2^2 k_{1x} - \mathrm{j}\, n_1^2 k_{2x}^{(\mathrm{i})}} = \exp\left(\mathrm{j}\, \varphi_{\mathrm{TM}}\right)$$

$$\varphi_{\mathrm{TM}} = 2 \arctan\left(\frac{n_1^2}{n_2^2} \frac{k_{2x}^{(\mathrm{i})}}{k_{1x}}\right) = 2 \arctan\left(\frac{n_1^2}{n_2^2} \frac{\sqrt{n_1^2 \sin^2 \vartheta_1 - n_2^2}}{n_1 \cos \vartheta_1}\right)$$

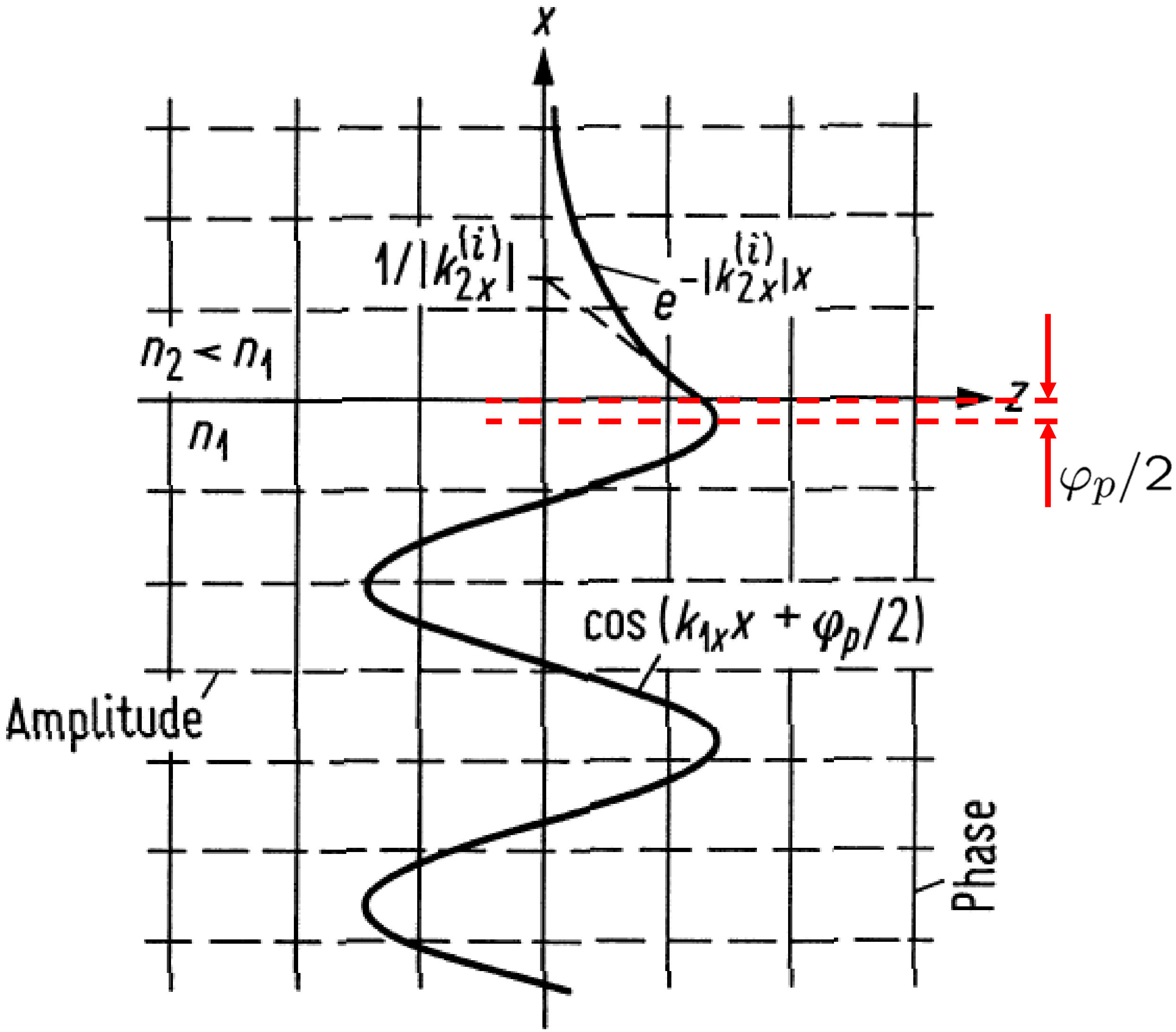
Combined:

$$R_{\mathrm{p}} = \exp\left(\mathrm{j}\, \varphi_p\right)$$

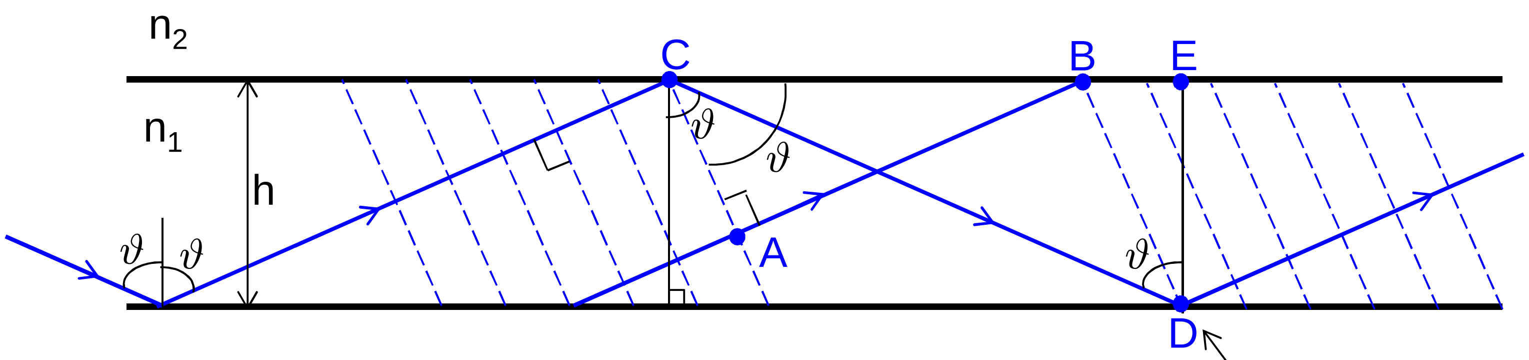
$$\varphi_{\mathrm{p}} = 2 \arctan\left(\sigma_{\mathrm{p}} \frac{\sqrt{n_1^2 \sin^2 \vartheta_1 - n_2^2}}{n_1 \cos \vartheta_1}\right)$$

$$\sigma_{\mathrm{p}} = \begin{cases} 1 & \text{for p=TE} \\ n_1^2/n_2^2 & \text{for p=TM} \end{cases}$$

Field distribution



Rays and plane waves in slab waveguides



Conditions for guidance of light:

- Necessary for total internal reflection: $\vartheta > \vartheta_{1T}$
- In addition: Only discrete angles of ϑ are permitted

Consider phase shifts along the rays AB and CD:

$$-n_1 k_0 \overline{\mathrm{AB}} = -n_1 k_0 \overline{\mathrm{CD}} + 2\varphi_{\mathrm{p}} + m\,2\pi, \qquad m \in \mathbb{N}$$

where $\overline{\mathrm{AB}} = (h \tan\vartheta - h/\tan\vartheta)\sin\vartheta$

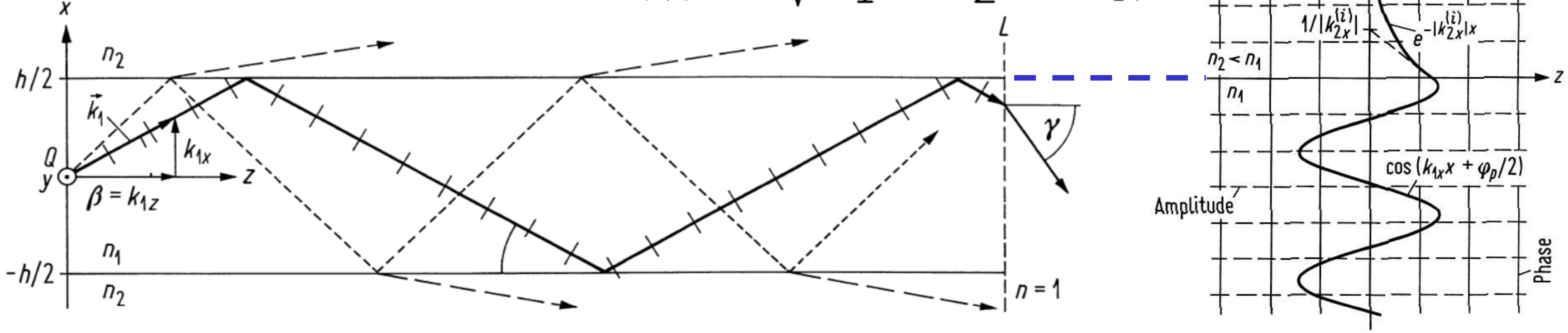
$$\overline{\mathrm{CD}} = h/\cos\vartheta$$

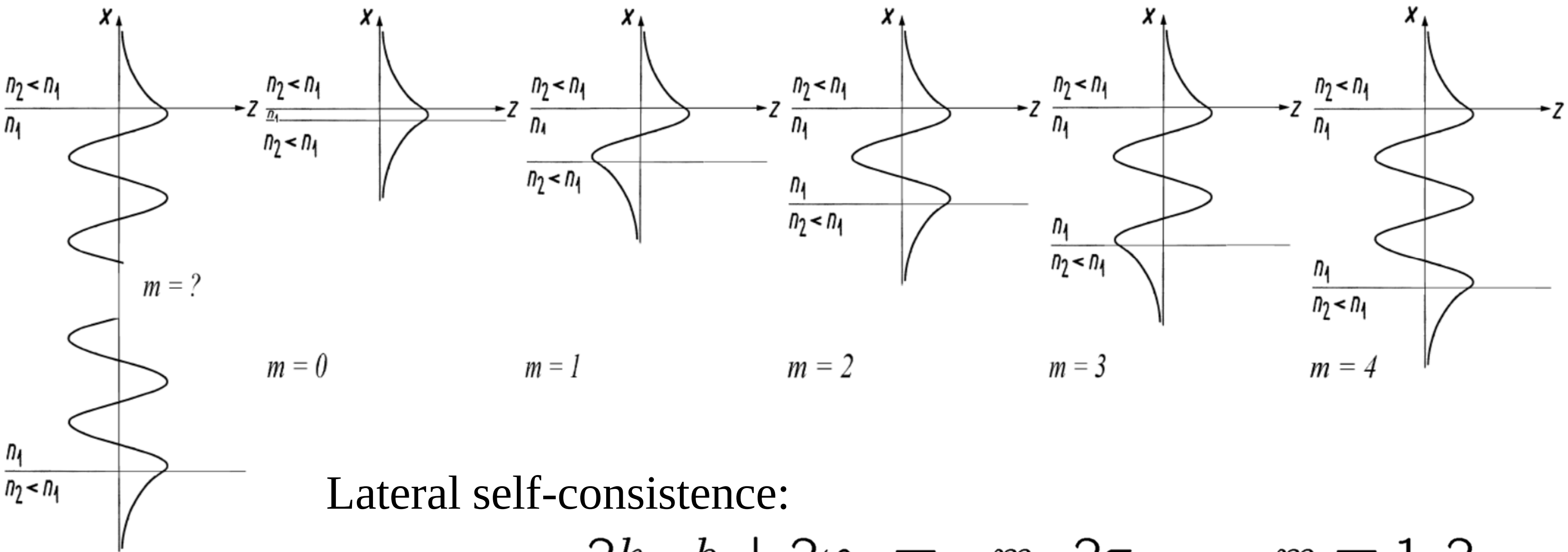
Permitted propagation angles ϑ defined by implicit equation:

$$2h\,n_1 k_0 \cos\vartheta = 2\varphi_{\mathrm{p}} + m\,2\pi \qquad m = 0, 1, \ldots$$

The slab waveguide – an intuitive approach

$$\sin(\gamma) = \sqrt{n_1^2 - n_2^2} = A_N$$





Lateral self-consistence:

$$-2k_{1x}h + 2\varphi_\mathrm{p} = -m \cdot 2\pi, \qquad m = 1, 2, \ldots$$

Eigenvalue equations for TE and TM modes

Consistency condition:

$$-2k_{1x}h + 2\varphi_\mathrm{p} = -m \cdot 2\pi, \qquad m_0 \in \mathbb{N}_0$$

Define:

$$u = \frac{h}{2}k_{1x} = \frac{h}{2}\sqrt{n_1^2 k_0^2 - \beta^2}$$

Transverse core phase constant

$$w = \frac{h}{2}k_{2x}^{(i)} = \frac{h}{2}\sqrt{\beta^2 - n_2^2 k_0^2}$$

Transverse cladding attenuation

$$V = \frac{h}{2}k_0\sqrt{n_1^2 - n_2^2} = \frac{h}{2}k_0 A_N = \sqrt{u^2 + w^2}$$

Normalized frequency

$$A_N = \sqrt{n_1^2 - n_2^2}$$

Numerical aperture

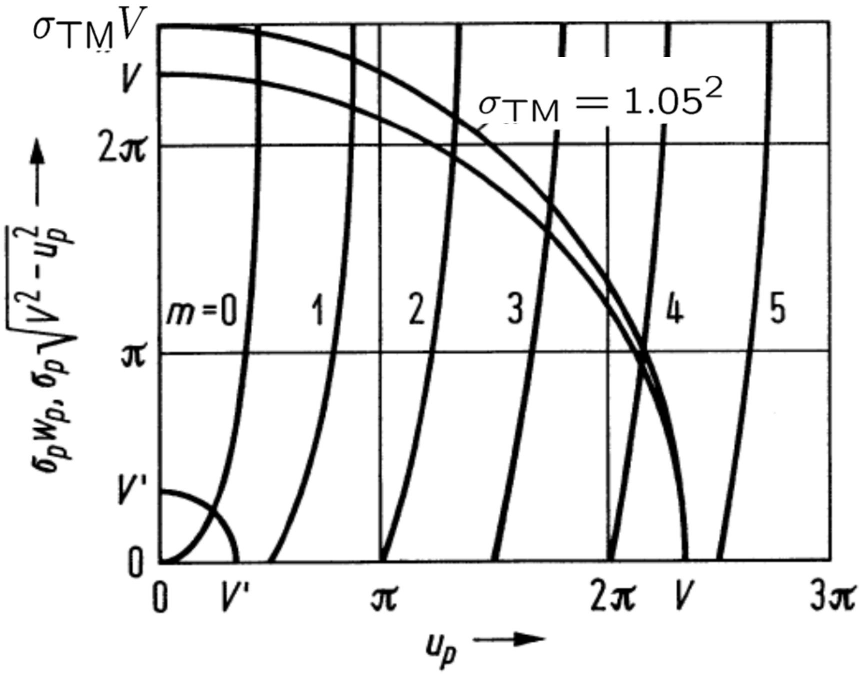
$$\beta = n_e k_0 = k_{1z} = k_{2z}$$

Mode propagation constant

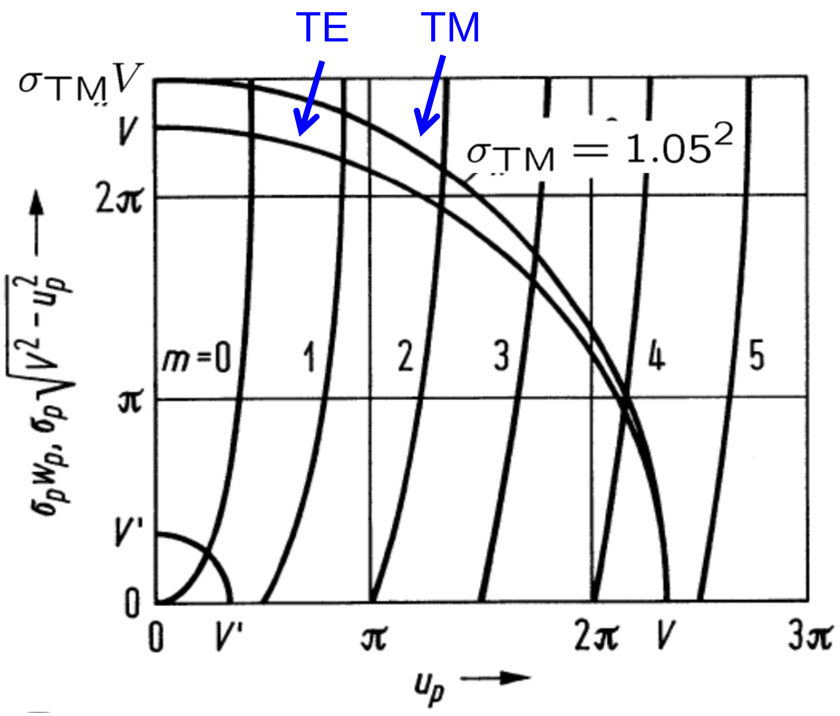
Eigenvalue equations for TE and TM:

$$u \tan\left(u - m\frac{\pi}{2}\right) = \sigma_\mathrm{p}\sqrt{V^2 - u^2}$$

$$\sigma_\mathrm{p} = \begin{cases} 1 & \text{for p=TE} \\ n_1^2/n_2^2 & \text{for p=TM} \end{cases}$$



Discussion: Modes of a slab waveguide



$$u = \frac{h}{2}\sqrt{n_1^2 k_0^2 - \beta^2}$$

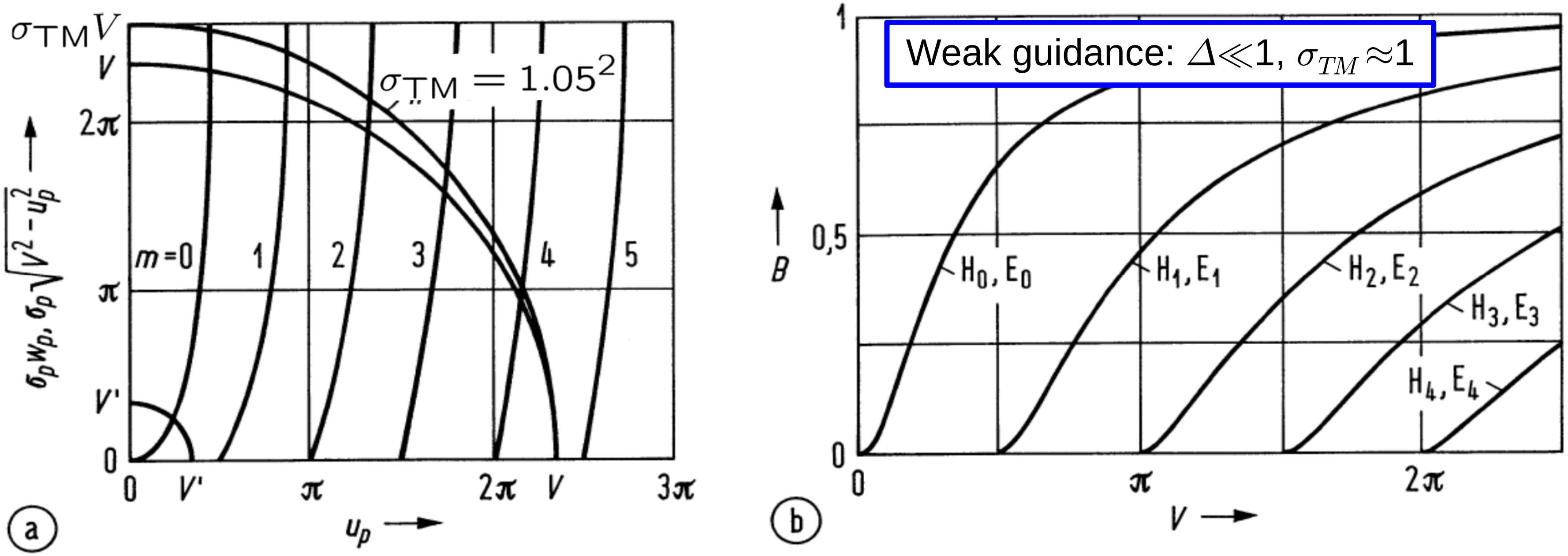
$$w = \frac{h}{2}\sqrt{\beta^2 - n_2^2 k_0^2}$$

$$V = \frac{h}{2} k_0 \sqrt{n_1^2 - n_2^2} = u^2 + w^2$$

$$\sigma_{TM} = n_1^2 / n_2^2$$

- The smaller *V*, the less modes are guided. Fundamental modes (*m* = 0) have the largest possible β (the smallest possible *u*).
- For V < π/2, there is only one guided TE and one guided TM mode. The waveguide is called single-mode.
- TE-modes have a always a larger β (smaller *u*) than the corresponding TM-modes.
- There is no lower cut-off frequency, i.e., the symmetric slab waveguide supports always at least one guided TE- and one guided TM-mode.
- For weak guidance, $n_1 \approx n_2$ and $\sigma_{TM} \approx 1$. The β-values for TM-modes and TE-modes approach each other asymptotically.

Dispersion relations



Mode propagation constant: $\beta = n_e k_0$

Transverse core phase constant: $u = \frac{h}{2}\sqrt{n_1^2 k_0^2 - \beta^2}$

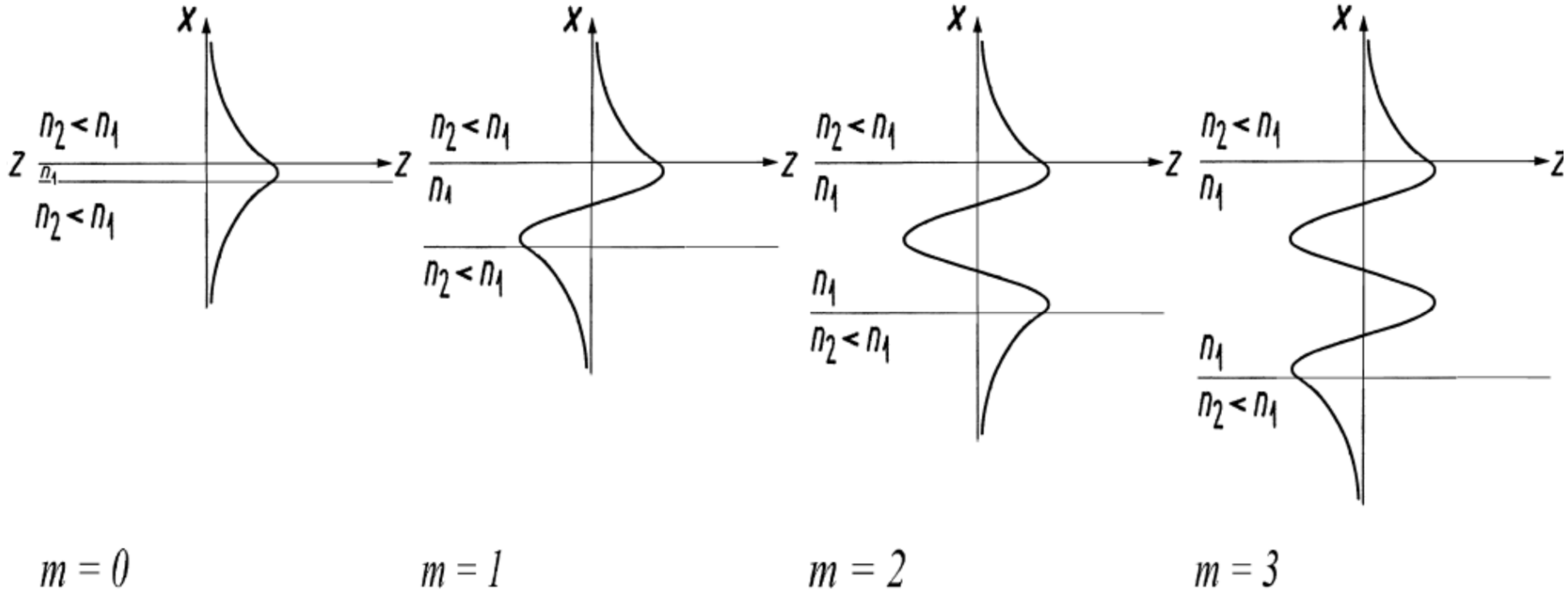
Transverse cladding attenuation: $w = \frac{h}{2}\sqrt{\beta^2 - n_2^2 k_0^2}$

Normalized frequency: $V = \frac{h}{2} k_0 \sqrt{n_1^2 - n_2^2} = \frac{h}{2} k_0 A_N = \sqrt{u^2 + w^2}$

Normalized propagation constant: $B = \frac{\beta^2 - n_2^2 k_0^2}{n_1^2 k_0^2 - n_2^2 k_0^2} = \frac{n_e^2 - n_2^2}{n_1^2 - n_2^2} = \frac{w^2}{V^2}, \qquad 0 < B < 1$

Note:
TE = “H-wave”
TM = “E-wave”

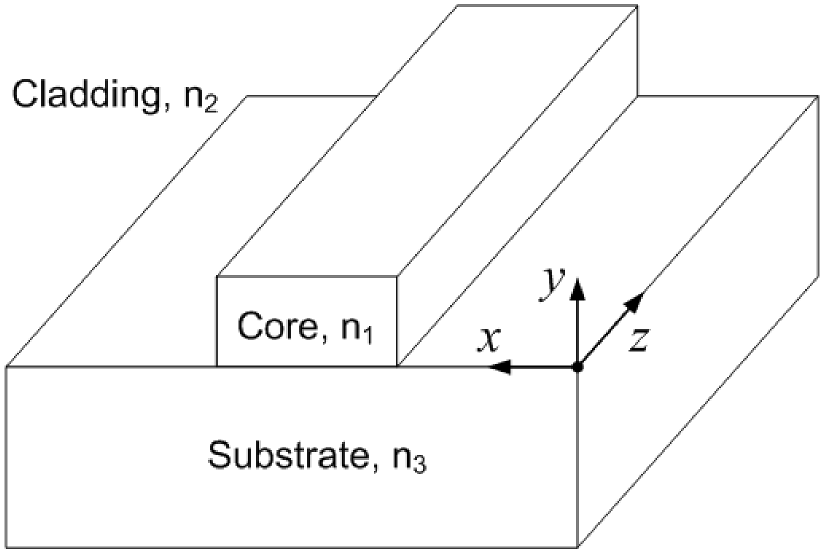
Guided modes of the dielectric slab waveguide



Field patterns of guided modes do not change during propagation along z!

Institute of Photonics
and Quantum Electronics

A more general approach: Modes of z-invariant optical structures



Lossless *z*-invariant dielectric structure ("lossless homogeneous waveguide"):

$$n = n(x,y)$$

where $\mathrm{Im}\{\underline{n}\} = 0$ throughout space.

Eigenmodes: A lossless homogenous waveguide features a set electromagnetic wave patterns which do not change their transverse shapes during propagation along z, so-called eigenmodes:

$$\underline{\mathbf{E}}(\mathbf{r},t) = \underline{\mathcal{E}}(x,y)\,\mathrm{exp}\left(\mathrm{j}\left(\omega t - \beta z\right)\right)$$

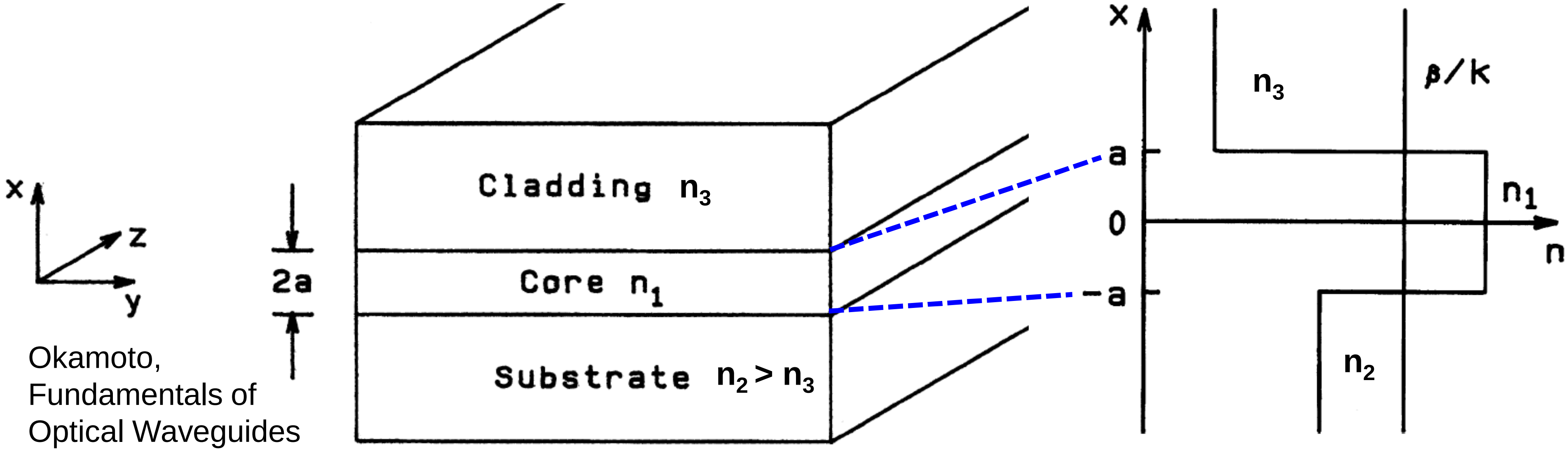
$$\underline{\mathbf{H}}(\mathbf{r},t) = \underline{\mathcal{H}}(x,y)\,\mathrm{exp}\left(\mathrm{j}\left(\omega t - \beta z\right)\right)$$

$$\beta = \frac{\omega}{c} n_e$$

Classification of eigenmodes:

- Real or imaginary β ?
- Mode fields confined to the waveguide core?

Propagating and evanescent eigenmodes



Okamoto, Fundamentals of Optical Waveguides

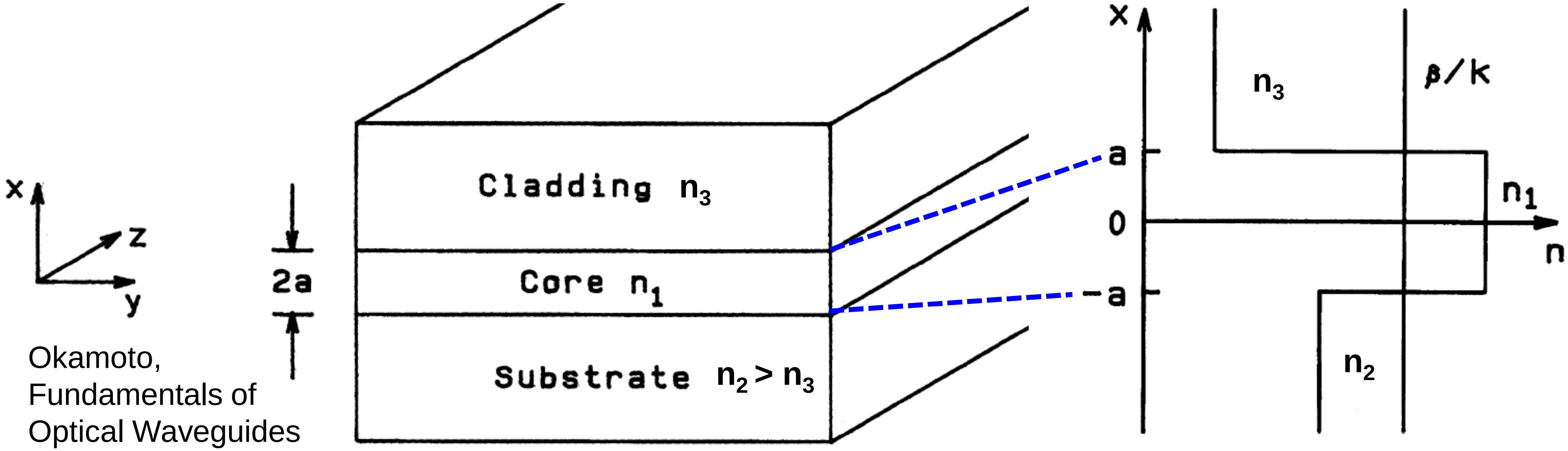
- Propagating eigenmodes are associated with a real propagation constant β and a real effective refractive index n_e, whereas for so-called evanescent eigenmodes, β and n_e are purely imaginary

- For propagating eigenmodes, the propagation constant obeys

$$|\beta| < n_1 k_0 \quad \text{for} \quad \beta \in \mathbb{R}$$

where n_1 is the maximum index in the waveguide cross section

Guided modes and radiation modes



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- For guided modes, the propagation constant β is real, and the fields are confined to the waveguide core,

$$\left.\begin{aligned} \underline{\mathcal{E}}(x,y) &\to 0 \text{ for } \left(x^2+y^2\right) \to \infty \\ \underline{\mathcal{H}}(x,y) &\to 0 \text{ for } \left(x^2+y^2\right) \to \infty \end{aligned}\right\} \text{in 3D!}$$

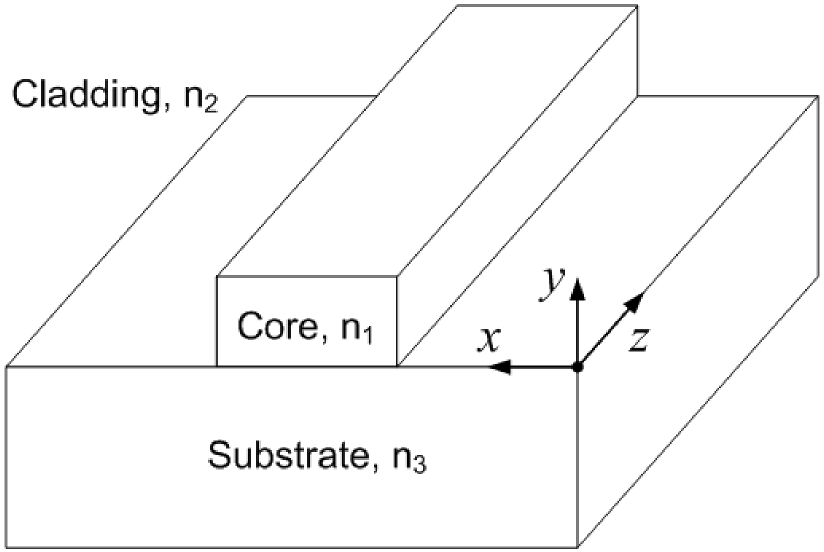
- Guided modes form a discrete set with propagation constants in the range

$$n_2 k_0 < |\beta| < n_1 k_0,$$

where n_2 denotes the maximum refractive index in the cladding region.

- For radiation modes, fields extend to infinity. Radiation modes form continuous sets and show an oscillatory behaviour to at least one side of the waveguide.

A more general approach: Modes of z-invariant optical structures



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Classification of eigenmodes:

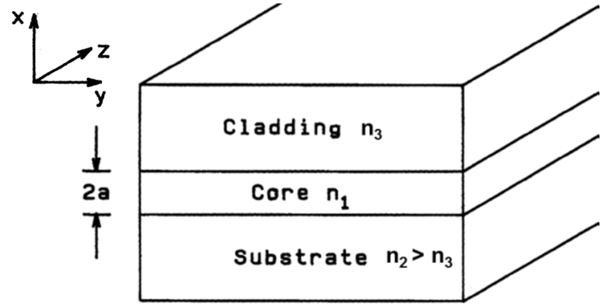
- Real or imaginary β ?
- Mode fields confined to the waveguide core?

Guided modes: Maxwell's equations for the mode fields

KIT – Karlsruhe Institute of Technology

Mode ansatz: $\underline{\mathbf{E}}(\mathbf{r},t) = \underline{\mathcal{E}}(x,y)\exp\left(\mathrm{j}\left(\omega t - \beta z\right)\right)$

$\underline{\mathbf{H}}(\mathbf{r},t) = \underline{\mathcal{H}}(x,y)\exp\left(\mathrm{j}\left(\omega t - \beta z\right)\right)$



$$\frac{\partial \underline{\mathcal{E}}_z}{\partial y} + \mathrm{j}\,\beta \underline{\mathcal{E}}_y = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_x$$

$$-\frac{\partial \underline{\mathcal{E}}_z}{\partial x} - \mathrm{j}\,\beta \underline{\mathcal{E}}_x = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_y$$

$$\frac{\partial \underline{\mathcal{E}}_y}{\partial x} - \frac{\partial \underline{\mathcal{E}}_x}{\partial y} = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_z$$

$$\frac{\partial \underline{\mathcal{H}}_z}{\partial y} + \mathrm{j}\,\beta \underline{\mathcal{H}}_y = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_x$$

$$-\frac{\partial \underline{\mathcal{H}}_z}{\partial x} - \mathrm{j}\,\beta \underline{\mathcal{H}}_x = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_y$$

$$\frac{\partial \underline{\mathcal{H}}_y}{\partial x} - \frac{\partial \underline{\mathcal{H}}_x}{\partial y} = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_z$$

⇒ **Slab waveguide:** $\frac{\partial}{\partial y} = 0$ ⇒

$$\mathrm{j}\,\beta \underline{\mathcal{E}}_y = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_x$$

$$-\frac{\partial \underline{\mathcal{E}}_z}{\partial x} - \mathrm{j}\,\beta \underline{\mathcal{E}}_x = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_y$$

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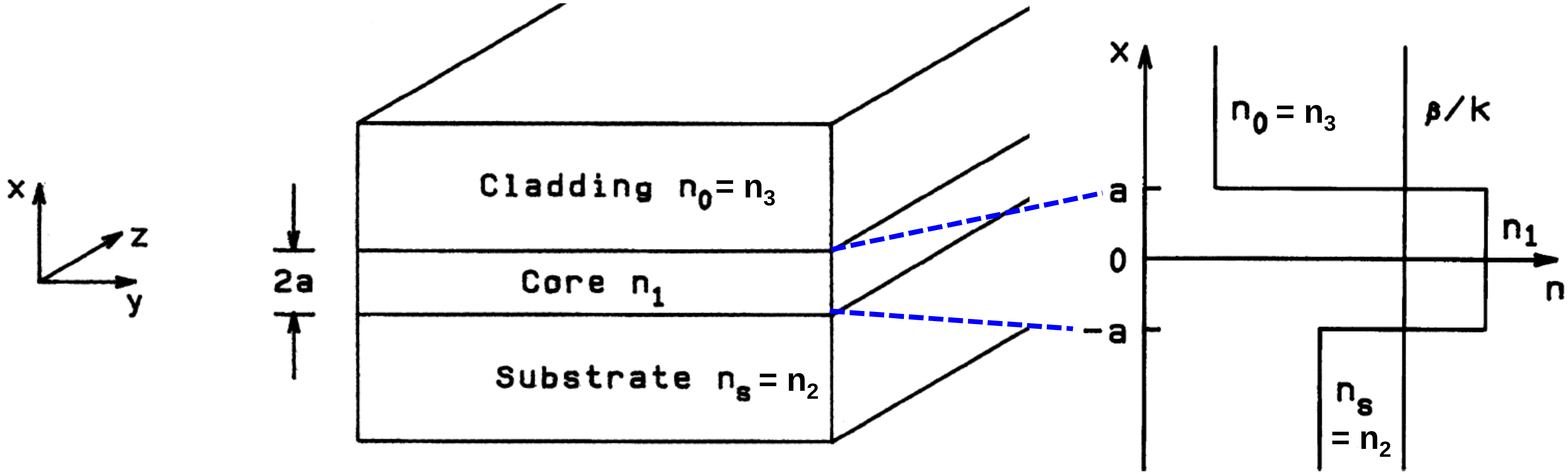
$$\frac{\partial \underline{\mathcal{H}}_y}{\partial x} = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_z$$

Separation of modes: TE: $\underline{\mathcal{E}}_y,\ \underline{\mathcal{H}}_x,\ \underline{\mathcal{H}}_z$ TM: $\underline{\mathcal{H}}_y,\ \underline{\mathcal{E}}_x,\ \underline{\mathcal{E}}_z$

Wave equation for the lateral mode fields



$$\frac{\partial^2 \underline{\mathcal{E}}_y(x)}{\partial x^2} + \left(\omega^2 \mu_0 \epsilon_0 n^2(x) - \beta^2\right) \underline{\mathcal{E}}_y(x) = 0 \qquad \textbf{TE}$$

$$n^2(x) \frac{\partial}{\partial x}\left(\frac{1}{n^2(x)} \frac{\partial \underline{\mathcal{H}}_y(x)}{\partial x}\right) + \left(\omega^2 \mu_0 \epsilon_0 n^2(x) - \beta^2\right) \underline{\mathcal{H}}_y(x) = 0 \qquad \textbf{TM}$$

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TE mode solution

Ansatz for E_y-component:

$$\underline{\mathcal{E}}_y(x) = \begin{cases} A\cos(k_{1x}x - \varphi) & \text{for } -a \le x \le a \\ A\cos(-k_{1x}a - \varphi)\exp\left(k_{2x}^{(i)}(x+a)\right) & \text{for } x < -a \\ A\cos(k_{1x}a - \varphi)\exp\left(-k_{3x}^{(i)}(x-a)\right) & \text{for } x > a \end{cases}$$

$$k_{1x} = \sqrt{n_1^2 k_0^2 - \beta^2}, \qquad k_{2x}^{(i)} = \sqrt{\beta^2 - n_2^2 k_0^2}, \qquad k_{3x}^{(i)} = \sqrt{\beta^2 - n_3^2 k_0^2}$$

Corresponding H_z-component:

$$\underline{\mathcal{H}}_z(x) = \frac{\mathrm{j}}{\omega\mu_0} \begin{cases} -k_{1x}A\sin(k_{1x}x - \varphi) & \text{for } -a \le x \le a \\ k_{2x}^{(i)}A\cos(-k_{1x}a - \varphi)\exp\left(k_{2x}^{(i)}(x+a)\right) & \text{for } x < -a \\ -k_{3x}^{(i)}A\cos(k_{1x}a - \varphi)\exp\left(-k_{3x}^{(i)}(x-a)\right) & \text{for } x > a \end{cases}$$

H_z must be continuous at $x = \pm a$

TE mode solution

Eigenvalue equations for β and φ:

$$\tan(u+\varphi) = \frac{w}{u}$$
$$\tan(u-\varphi) = \frac{w'}{u}$$

$$u = \frac{1}{2}\arctan\left(\frac{w}{u}\right) + \frac{1}{2}\arctan\left(\frac{w'}{u}\right) + \frac{m\pi}{2}$$
$$\varphi = \frac{1}{2}\arctan\left(\frac{w}{u}\right) - \frac{1}{2}\arctan\left(\frac{w'}{u}\right) + \frac{m\pi}{2}$$

$$u = k_{1x}a = a\sqrt{n_1^2k_0^2 - \beta^2};\;\; w = k_{2x}^{(i)}a = a\sqrt{\beta^2 - n_2^2k_0^2};\;\; w' = k_{3x}^{(i)}a = a\sqrt{\beta^2 - n_3^2k_0^2}$$

Numerical solution of eigenvalue equation: New parameters *V*, γ, *B*

Normalized frequency $$V = ak_0\sqrt{n_1^2 - n_2^2}$$

Asymmetry parameter $$\gamma = \frac{n_2^2 - n_3^2}{n_1^2 - n_2^2}$$

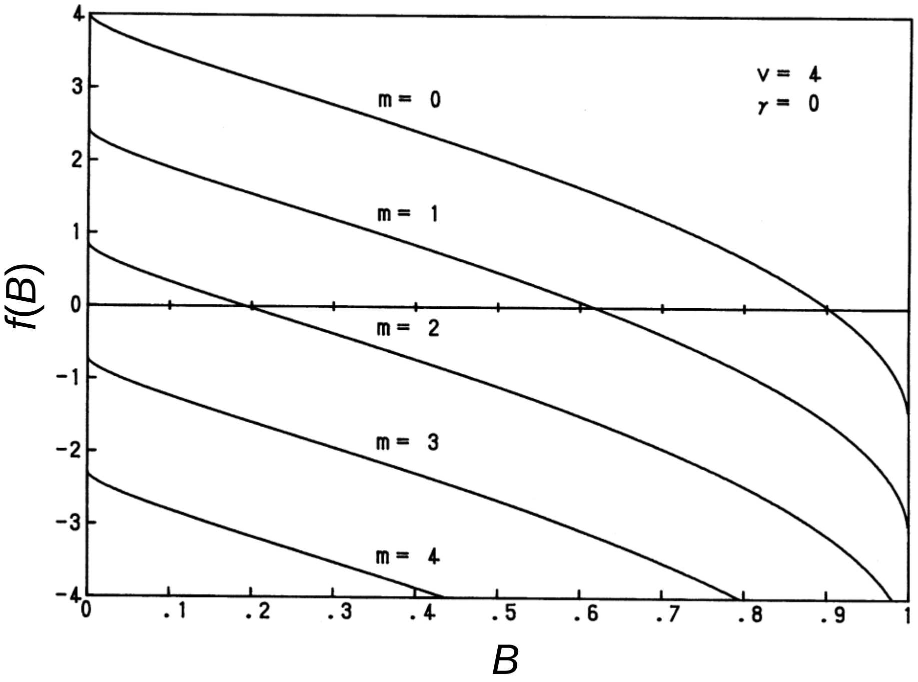
Normalized propagation constant $$B = \frac{\beta^2 - n_2^2k_0^2}{n_1^2k_0^2 - n_2^2k_0^2} = \frac{n_e^2 - n_2^2}{n_1^2 - n_2^2}$$

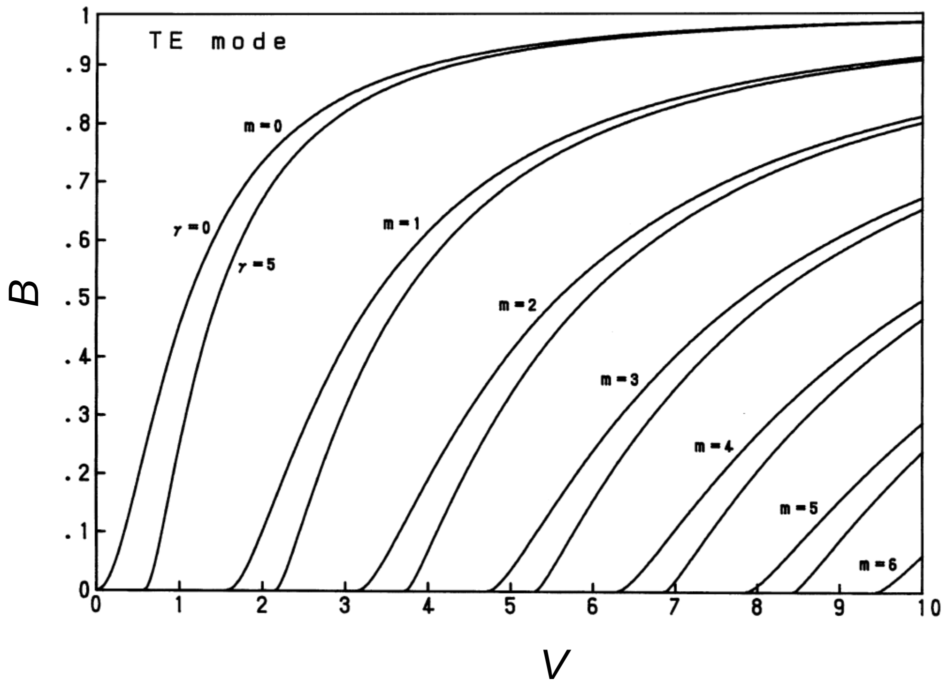
$$u = V\sqrt{1-B};\;\; w = V\sqrt{B};\;\; w' = V\sqrt{\gamma + B}$$

...back to channel waveguide analysis

Numerical solution of eigenvalue equation for TE

$$\underbrace{V\sqrt{1-B}-\frac{1}{2}\arctan\left(\sqrt{\frac{B}{1-B}}\right)-\frac{1}{2}\arctan\left(\sqrt{\frac{\gamma+B}{1-B}}\right)-\frac{m\pi}{2}}_{f(B)}=0$$





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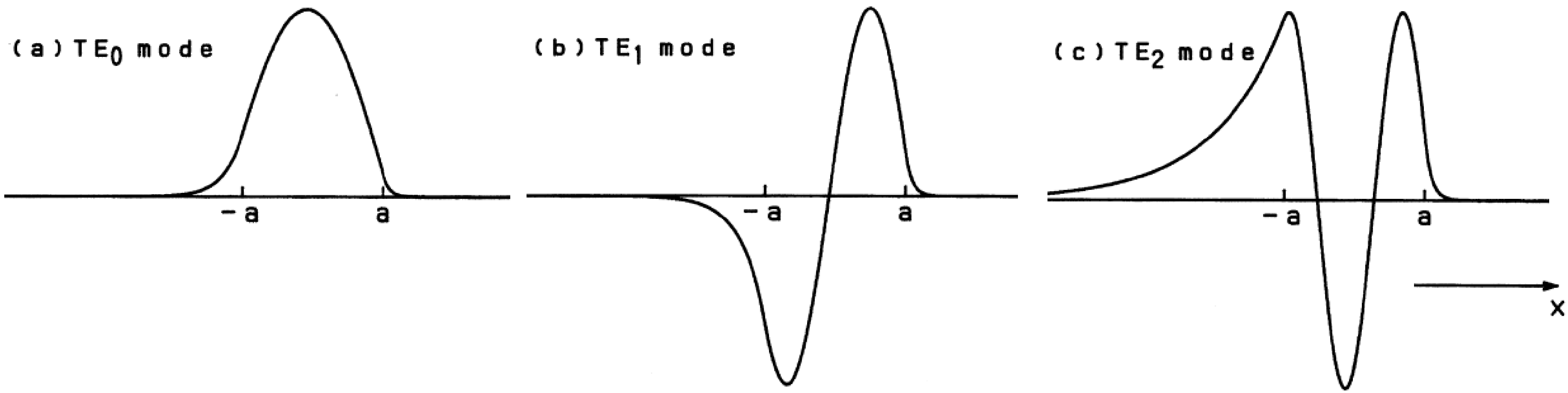
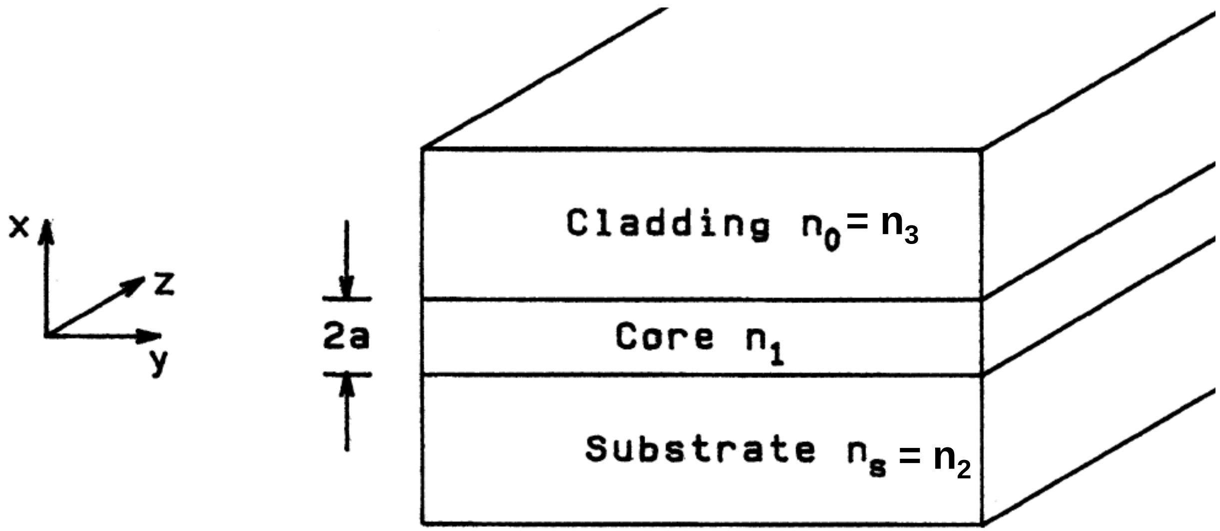
Slab waveguide



$n_1 = 3.38$
$n_2 = 3.17$
$n_3 = 1$

$\gamma = 6.6$

$V = 4$



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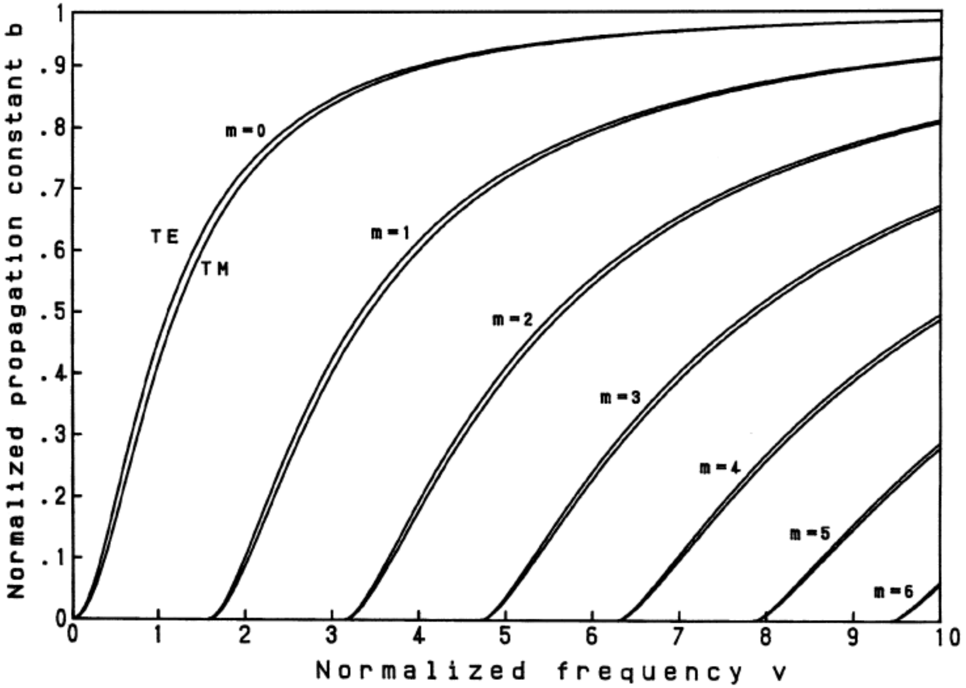
Eigenvalue equation for TM modes

Similar derivation as in TE case:

$$u = \frac{1}{2}\arctan\left(\frac{n_1^2}{n_2^2}\frac{w}{u}\right) + \frac{1}{2}\arctan\left(\frac{n_1^2}{n_2^2}\frac{w'}{u}\right) + \frac{m\pi}{2}$$

$$\underbrace{V\sqrt{1-B} - \frac{1}{2}\arctan\left(\frac{n_1^2}{n_2^2}\sqrt{\frac{B}{1-B}}\right) - \frac{1}{2}\arctan\left(\frac{n_1^2}{n_3^2}\sqrt{\frac{\gamma+B}{1-B}}\right) - \frac{m\pi}{2}}_{f(n_1,n_2,n_3,m,V,B)} = 0$$



$n_1 = 3.38$
$n_2 = n_3 = 3.17$

Propagation constant for the TM mode is smaller than for the TE mode, i.e., TE is better confined to the core.

Okamoto, Fundamentals of Optical Waveguides

Summary: Calculating guided modes of slab waveguides

- Frequency ω
- Waveguide parameters n_1, n_2, n_3, a

→

- Normalized frequency V
- Asymmetry parameter γ

Numerical solution of eigenvalue equation →

- Normalized propagation constans B_m for different modes (mode index m)

→

- Transverse phase constant u_m / cladding attenuation w_m, w'_m
- φ_m, β_m, $k_{1x,m}$, $k_{2x,m}^{(i)}$, $k_{3x.m}^{(i)}$

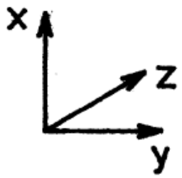
→

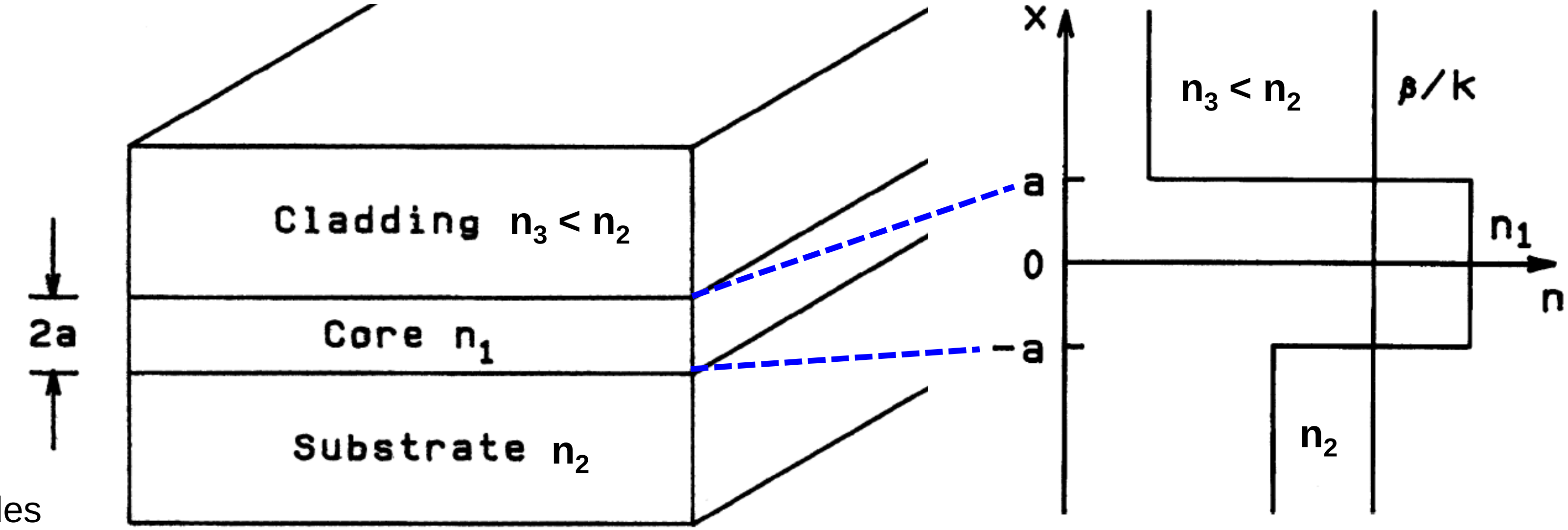
- Field plots

Repeat for different frequencies ω →

- Dispersion relation for each mode: $\beta_m = \beta_m(\omega)$,

Radiation modes of an asymmetric slab waveguide





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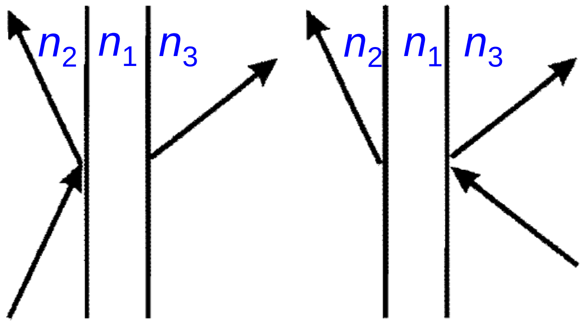
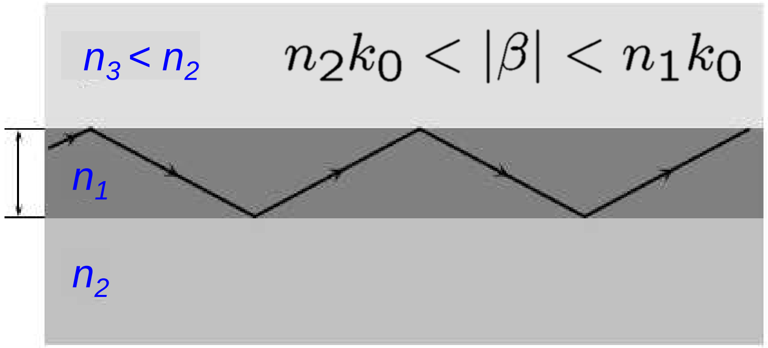


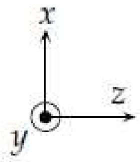
Propagating radiation modes can be thought of as plane waves impinging of the waveguide structure from outside. These modes show an oscillatory behaviour to at least one side of the waveguide structure.

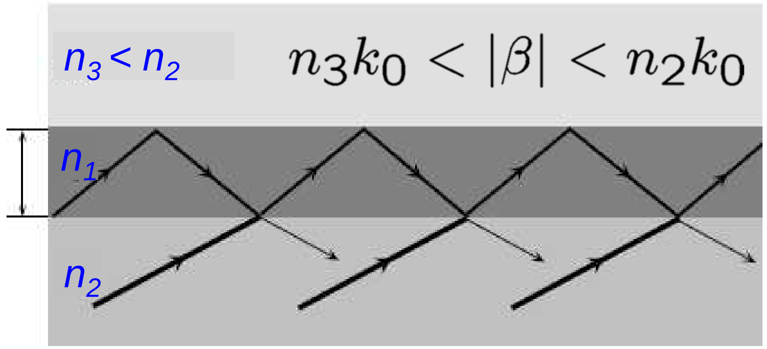
For asymmetric slab waveguides:

- **Substrate mode:** Oscillatory behaviour in the substrate only, evanescent in the cladding
$$n_3 k_0 < |\beta| < n_2 k_0$$
- **Cover mode:** Oscillatory behaviour in both substrate and cover:
$$|\beta| < n_3 k_0$$

Radiation modes of an asymmetric slab waveguide

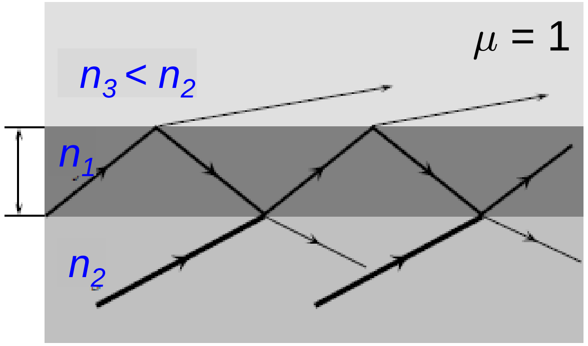


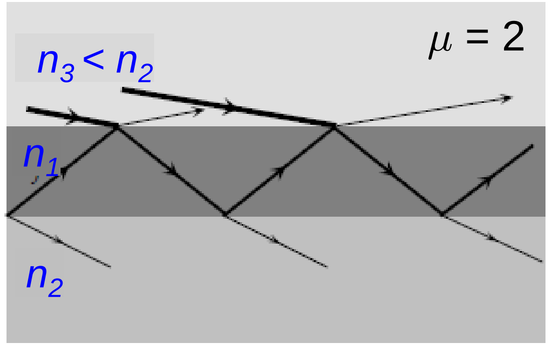




Guided mode

Radiation mode („substrate mode“)





Radiation mode („cover modes“)

$$|\beta| < n_3 k_0$$

Chen, Guided Wave Optics

Note: The external plane waves associated with the radiation modes can have any propagation direction.

$\Rightarrow$ In contrast to guided modes, propagating radiation modes form continuous sets with propagation constants

$$|\beta| < n_2 k_0 \quad \text{for} \quad \beta \in \mathbb{R} \text{ (propagating eigenmode)}$$

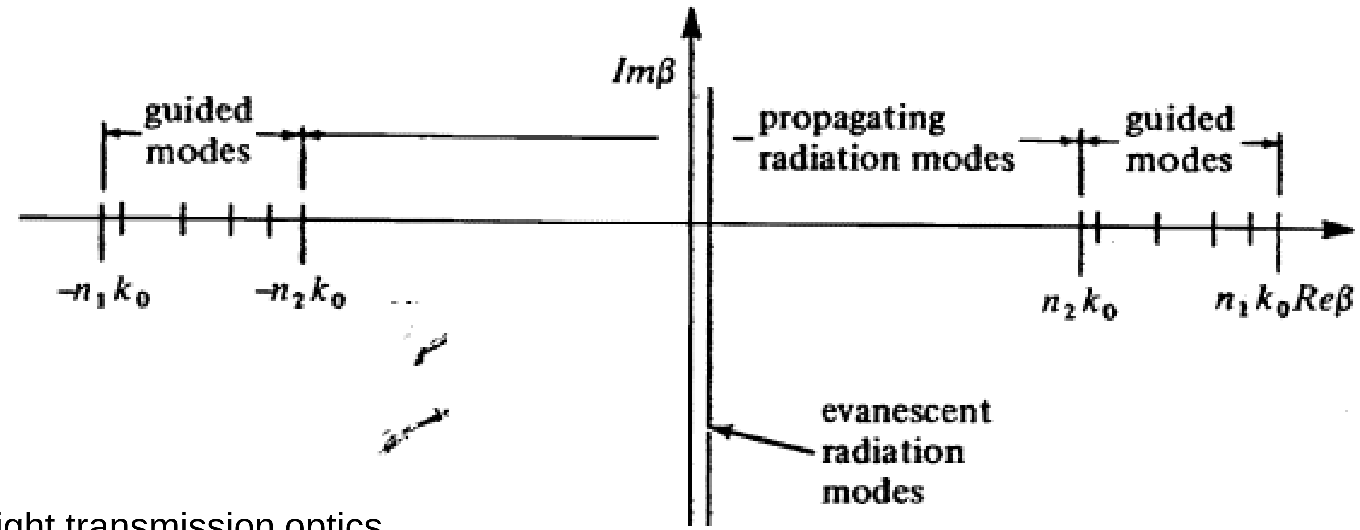
Evanescent radiation modes

Evanescent radiation modes decay exponentially in the direction of propagation. These modes have purely imaginary propagation constants β and large wave vector components in the transverse direction

$$n_3 k_0 < k_x < \infty$$

Such modes are, e.g., needed to describe the fine structure of the field in the vicinity of a sub-wavelength waveguide imperfection.

Propagation constants β for guided modes and radiation modes in the complex plane (symmetric slab waveguide):

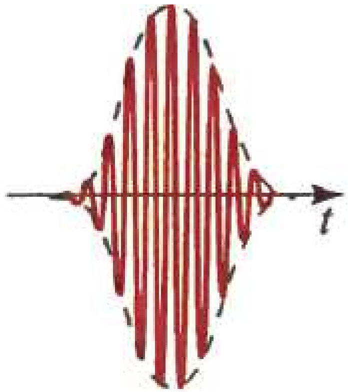


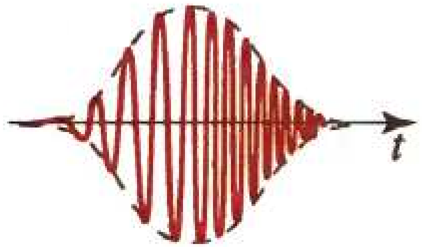
Marcuse, Light transmission optics

Institute of Photonics and Quantum Electronics

Signal propagation in dispersive waveguide

Propagation in a single waveguide mode: Same description as for dispersive homogeneous medium.





Slab waveguide: $\beta = \beta(\omega)$

$$\underline{a}\,(0,t) = \underline{A}\,(0,t)\exp\left(\mathrm{j}\,\omega_c t\right) \quad \xrightarrow{\;z\;} \quad \underline{a}\,(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{\tilde{A}}\,(0,\omega-\omega_c)\,\mathrm{e}^{-\mathrm{j}\,\beta(\omega) z}\,\mathrm{e}^{\mathrm{j}\,\omega t}\,\mathrm{d}\omega$$

Propagation: $\mathrm{e}^{-\mathrm{j}\,\beta(\omega) z}$

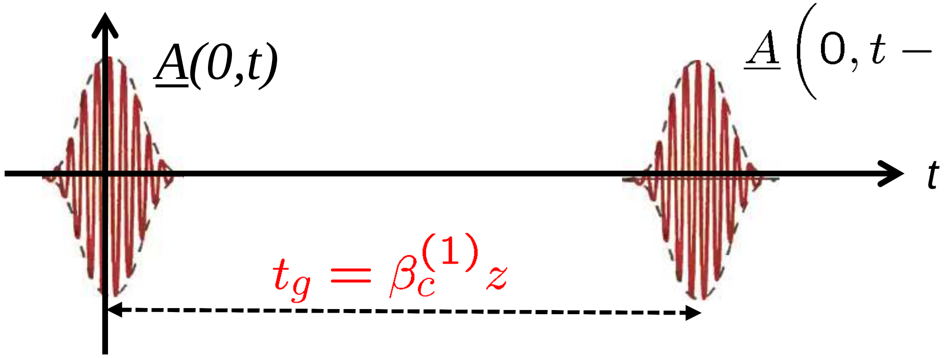
$$\underline{\tilde{a}}\,(0,\omega) = \underline{\tilde{A}}\,(0,\omega-\omega_c) \quad \longrightarrow \quad \underline{\tilde{a}}\,(z,\omega) = \underline{\tilde{A}}\,(0,\omega-\omega_c)\,\mathrm{e}^{-\mathrm{j}\,\beta(\omega) z}$$

Taylor expansion of mode propagation constant:

$$\beta(\omega) = \frac{\omega}{c} n_e(\omega) \approx \beta_c^{(0)} + (\omega-\omega_c)\beta_c^{(1)} + \frac{(\omega-\omega_c)^2}{2!}\beta_c^{(2)} + \frac{(\omega-\omega_c)^3}{3!}\beta_c^{(3)} + \ldots$$

where $\beta_c^{(i)} = \left.\frac{\mathrm{d}^i \beta(\omega)}{\mathrm{d}\omega^i}\right|_{\omega=\omega_c}$

Propagation in a single waveguide mode: Group and phase delay



$$\underline{A}\left(0, t - \beta_c^{(1)} z\right) \mathrm{e}^{\mathrm{j}\left(\omega_c t - \beta_c^{(0)} z\right)} .$$

Group delay ($\beta_c^{(1)} z$), Phase delay ($\beta_c^{(0)} z$)

Group delay and group velocity of signal envelope:

$$t_g = \frac{z}{v_g} = \beta_c^{(1)} z$$

$$v_g = \frac{1}{\beta_c^{(1)}} = \frac{c}{n_{eg}}$$

Effective group refractive index:

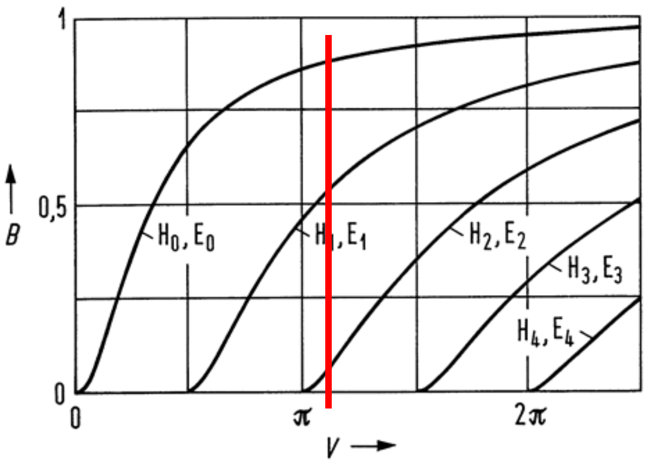
$$n_{eg}(\omega_c) = n_e(\omega_c) + \omega_c \left.\frac{\mathrm{d}n_e(\omega)}{\mathrm{d}\omega}\right|_{\omega=\omega_c}$$

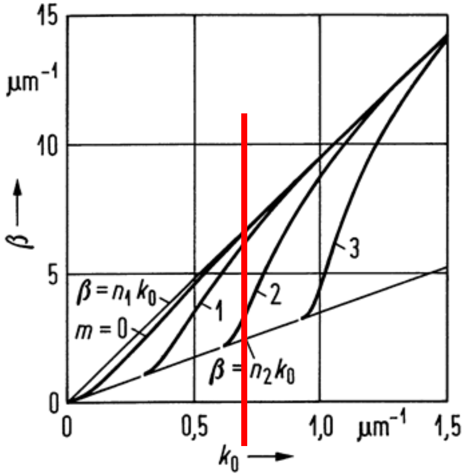
$$= n_e(\lambda_c) - \lambda_c \left.\frac{\mathrm{d}n_e(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_c}$$

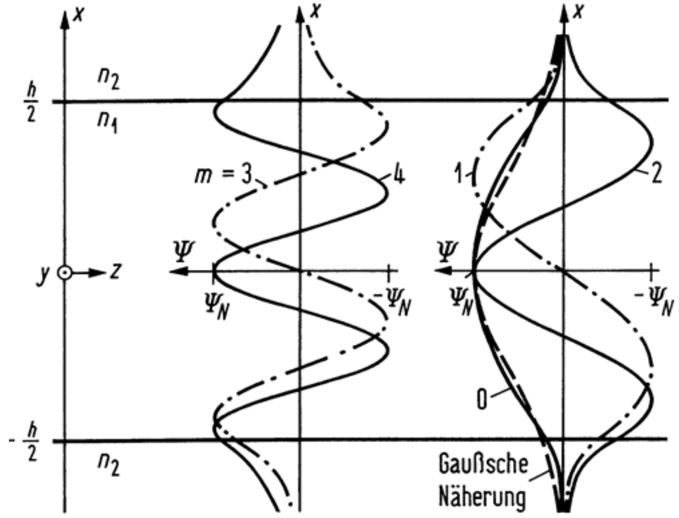
Intermodal dispersion

- Multimode propagation: Signal can propagate in different modes, all featuring different group velocities
  $\Rightarrow$ Intermodal dispersion / (Inter-)Modendispersion

Example: Symmetric slab waveguide



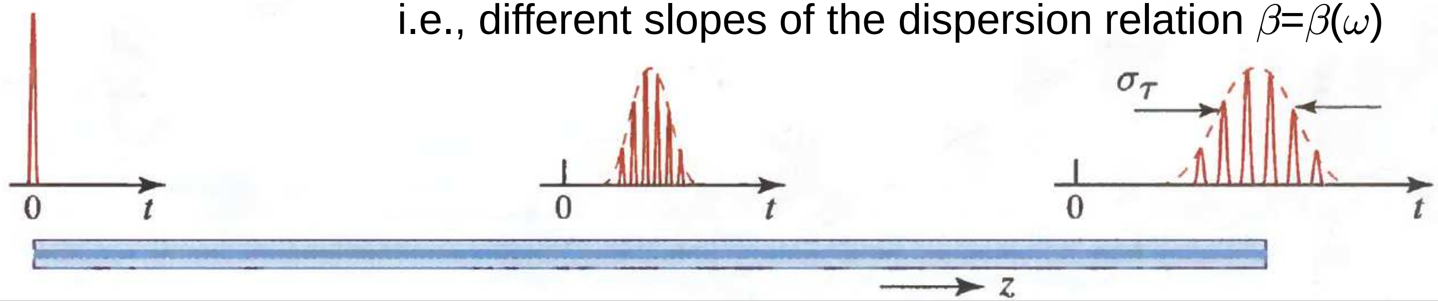




$$t_{g,m} = \beta_m^{(1)}(\omega_c)\,L$$

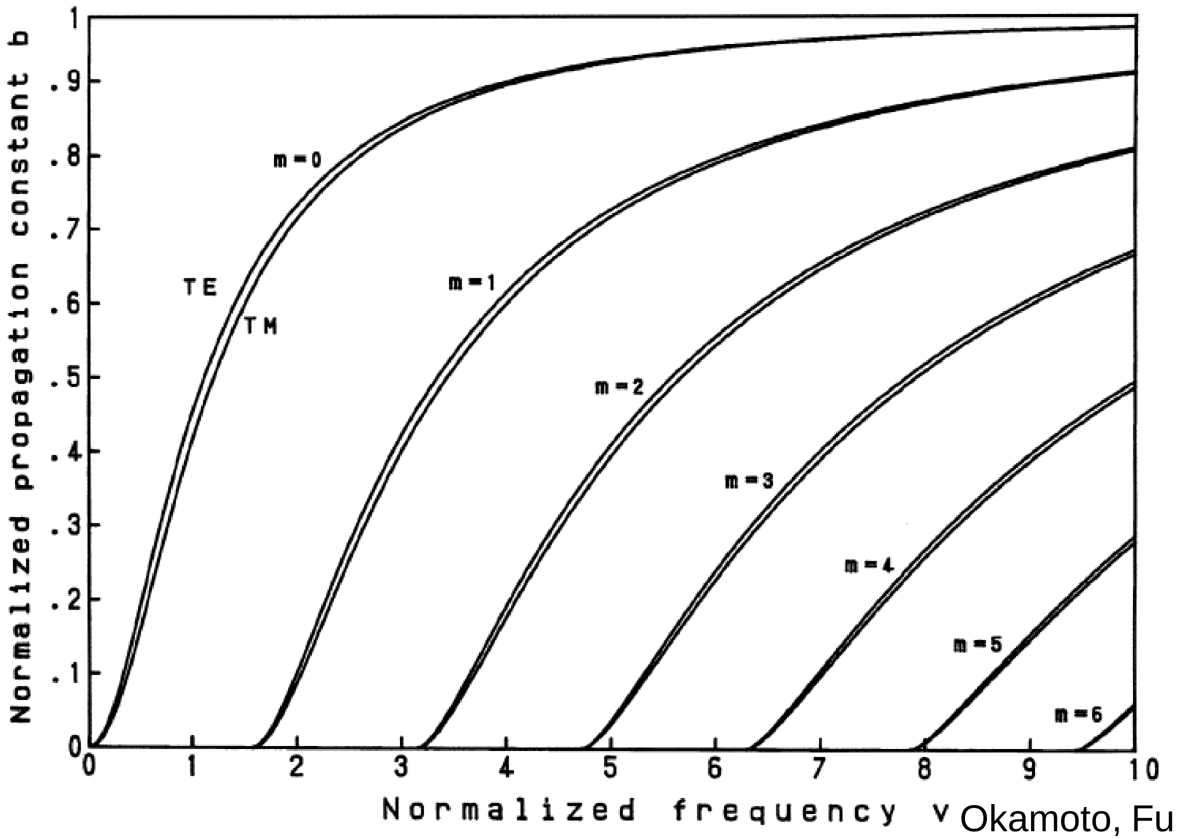
m = mode index

Different modes experience different group delays, i.e., different slopes of the dispersion relation $\beta=\beta(\omega)$



Polarization mode dispersion

- For “single-mode” waveguides: Still, two polarizations can propagate; the fundamental TE and TM mode have different dispersion relations
  $\Rightarrow$ Polarization mode dispersion (PMD)
  Polarisationsmodendispersion



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Intramodal dispersion

- If only one polarization is excited, the wavelength-dependence of the group velocity remains
  ⇒ Intramodal dispersion / Intramodendispersion
  = Group velocity dispersion (GVD) / Gruppengeschwindigkeitsdispersion
  = Chromatic dispersion / Chromatische Dispersion

$$\frac{\triangle t_g}{z} = C_\lambda \, \Delta\lambda_c + D_\lambda \, \Delta\lambda_c^2, \qquad C_\lambda = -\frac{2\pi c}{\lambda^2}\beta_c^{(2)}$$

Two contributions to chromatic dispersion:

1. Material dispersion M_λ: Frequency dependence of the material's refractive indices

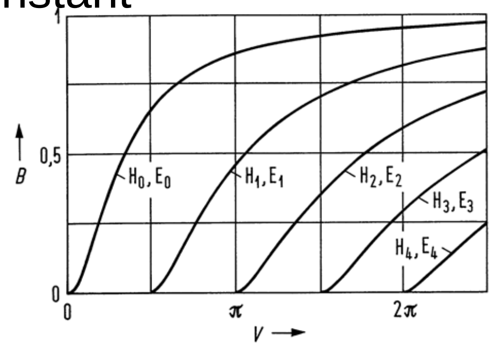
$$M_\lambda = \frac{1}{c}\frac{\mathrm{d}n_g(\lambda)}{\mathrm{d}\lambda}.$$

2. Waveguide dispersion W_λ: Boundary conditions in waveguide lead to frequency dependence of the modal propagation constant

For weakly guided modes:

$$C_\lambda = M_\lambda + W_\lambda$$

$$M_\lambda = \frac{1}{c}\frac{\mathrm{d}n_g(\lambda)}{\mathrm{d}\lambda}, \qquad W_\lambda = -\frac{n_{1g}-n_{2g}}{c\lambda}\underbrace{V\frac{\mathrm{d}^2(VB)}{\mathrm{d}V^2}}_{\text{dispersion factor}}$$



Interplay of material and waveguide dispersion

Compare two different symmetric slab waveguides with $n_2 = n_3 = 1.45$ and $n_1 = n_2 + \delta n$:

Waveguide 1: $\delta n = 0.005$; $2a = 4\ \mu\text{m}$
Waveguide 2: $\delta n = 0.005 \times 10$; $2a = 4\ \mu\text{m}\,/\sqrt{10}$

For both waveguides, a certain real frequency f corresponds to the same normalized frequency V.

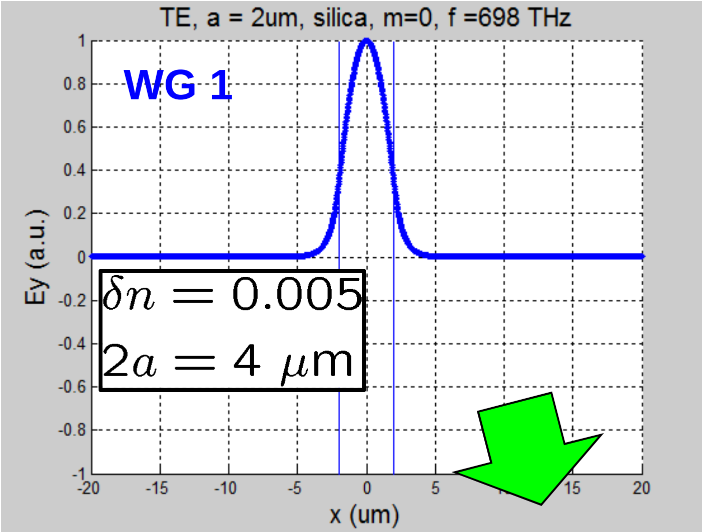
Eigenvalue equations: γ = 0 (symmetric waveguide)

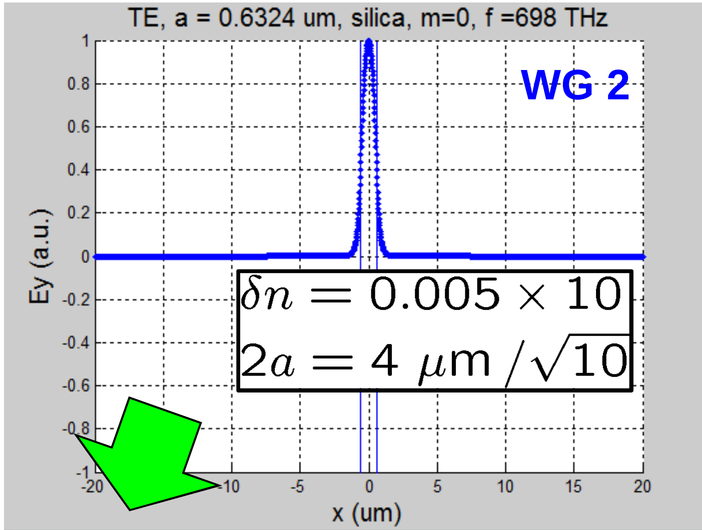
TE:
$$\underbrace{V\sqrt{1-B} - \frac{1}{2}\arctan\left(\sqrt{\frac{B}{1-B}}\right) - \frac{1}{2}\arctan\left(\sqrt{\frac{\gamma+B}{1-B}}\right) - \frac{m\pi}{2}}_{f(m,V,B)} = 0$$

TM:
$$\underbrace{V\sqrt{1-B} - \frac{1}{2}\arctan\left(\frac{n_1^2}{n_2^2}\sqrt{\frac{B}{1-B}}\right) - \frac{1}{2}\arctan\left(\frac{n_1^2}{n_3^2}\sqrt{\frac{\gamma+B}{1-B}}\right) - \frac{m\pi}{2}}_{f(n_1,n_2,n_3,m,V,B)} = 0$$

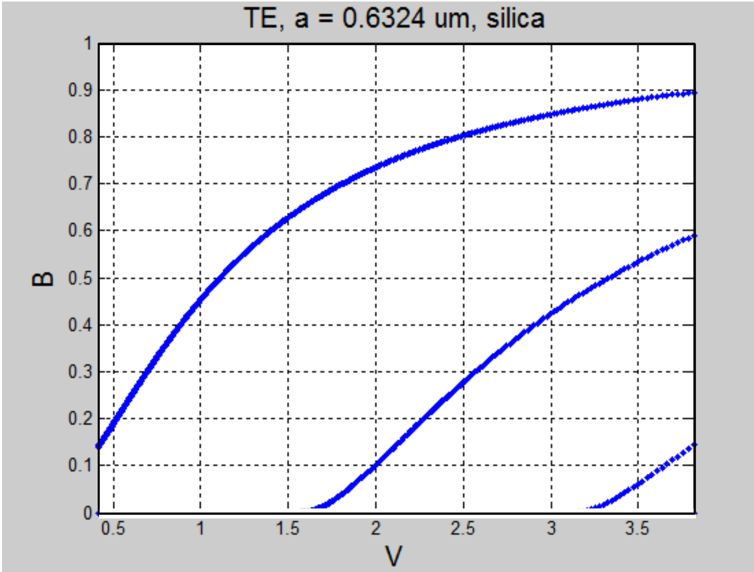
$\Rightarrow$ **Identical (nearly identical) normalized propagation constants for TE (TM) in both waveguides!**

Example (see problem set): Interplay of material and waveguide dispersion





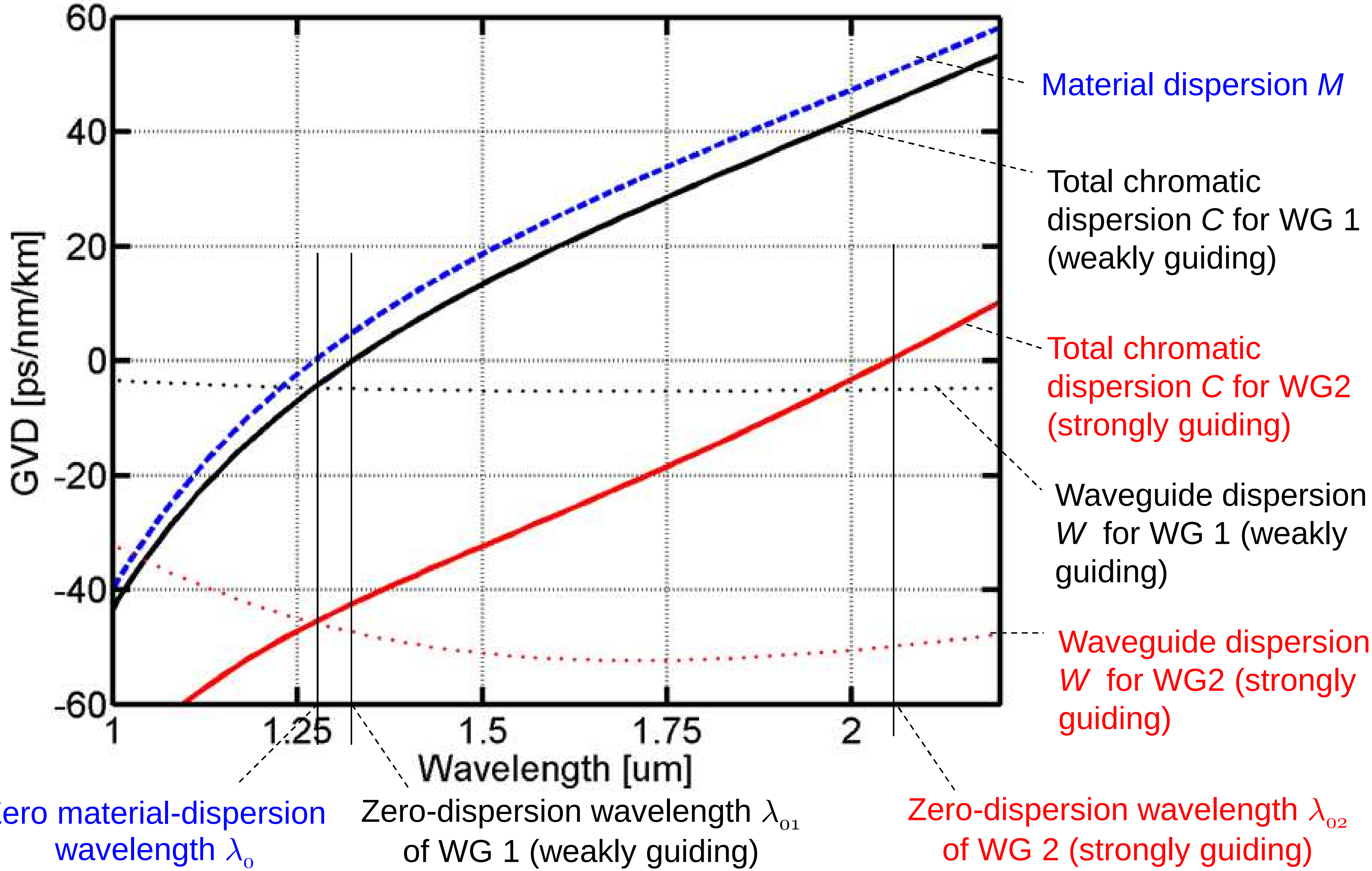
$$V = ak_0\sqrt{n_1^2 - n_2^2}$$
$$\approx ak_0\sqrt{2n_2\delta n}$$



$\Rightarrow$ Nearly identical normalized dispersion relations for both waveguides!

But: Dispersion characteristics differ quite significantly!

Interplay of material and waveguide dispersion



A Short Visit to Plasmonics

Metal-clad slab waveguide

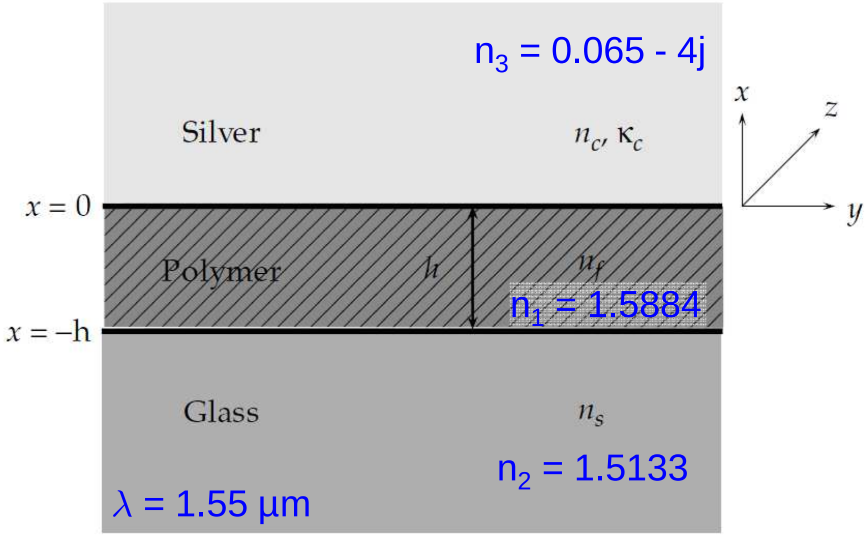


TABLE 4.1 Refractive indices and extinction coefficients of selected metals and semiconductors

Material	λ (µm)	n	κ	$\varepsilon_r - j\varepsilon_i = (n - j\kappa)^2$	F
Au	0.633	0.17	3.0	−8.97−j1.02	
	0.653	0.166	3.15	−9.89−j1.05	
	1.55	0.550	11.5	−132−j12.6	
Ag	0.633	0.065	3.9	−15.2−j0.507	
	0.653	0.140	4.15	−17.2−j1.16	
	1.55	0.514	10.8	−116−j11.1	
Cu	0.633	0.14	3.15	−9.91−j0.88	
	0.653	0.214	3.67	−13.4−j1.57	
	1.55	0.606	8.26	−67.9−j10.0	
Al	0.633	1.2	7	−47.56−j16.8	
	0.653	1.49	7.82	−58.9−j23.3	
	1.55	1.44	16.0	−254−j46.1	
Cr	0.633	3.19	2.26	+5.07−j14.4	
	1.590	4.13	5.03	−8.24−j41.5	
Ge	0.633	4.5	1.7	+17.4−j15.3	
	0.653	5.294	0.638	+27.6−j6.76	
	1.55	4.275	0.00567	+18.3−j0.049	
GaAs	0.633	3.856	0.196	+14.8−j1.51	
	0.653	3.826	0.179	+14.6−j1.37	
	1.55	3.3737	—	+11.4	
Si	0.633	3.882	0.019	+15.07−j0.148	
	0.653	3.847	0.016	+15.0−j0.123	
	1.532	3.4784	—	+12.1	

Susceptibility of metals

Ideal metal:

$$\epsilon_r = 1 - \frac{\omega_p^2}{\omega^2}$$

Real metal:

- Bound + free charges
- Damping of electron motion

=> Komplex $\underline{\epsilon}_r$:

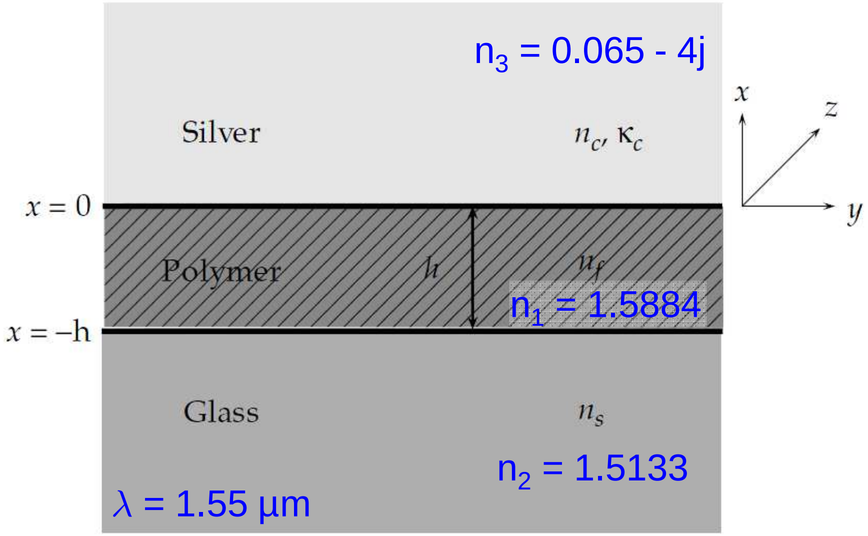
$$\underline{\epsilon}_r = \epsilon_r - \mathrm{j}\epsilon_{ri}$$

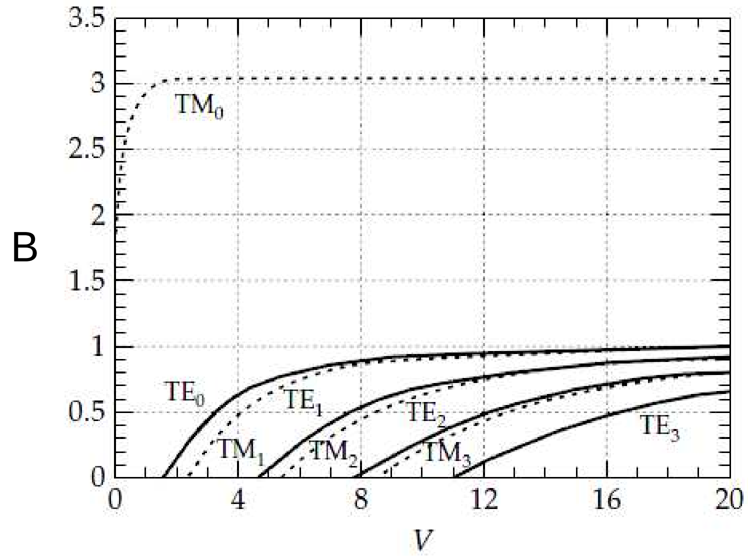
$$\underline{n} = n - \mathrm{j}n_i = n - \mathrm{j}\kappa$$

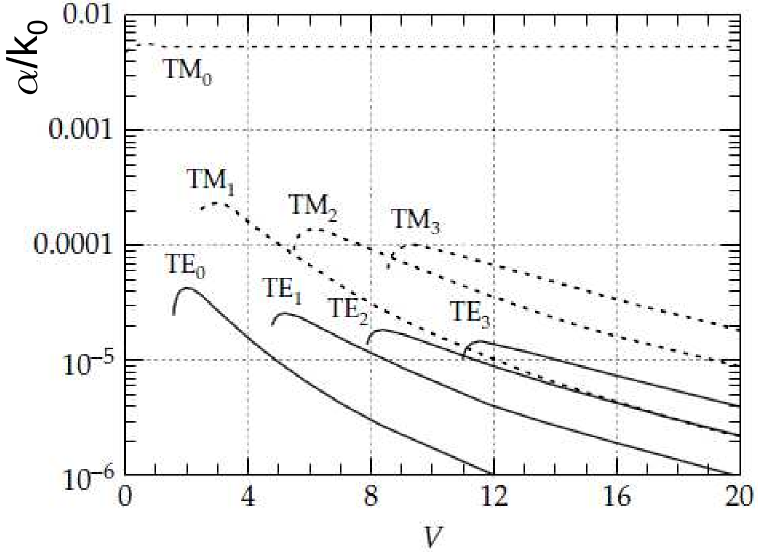
Extinction coefficient

Chen, Guided Wave Optics

Modes of metal-clad slab waveguide







For most modes:

- B-V-curves have essentially same shape as for dielectric slab waveguide
- TM-modes have larger loss than TE-modes
- Loss coefficient peaks slightly above the cutoff frequency and then decreases with frequency

Exception: TM_0-mode

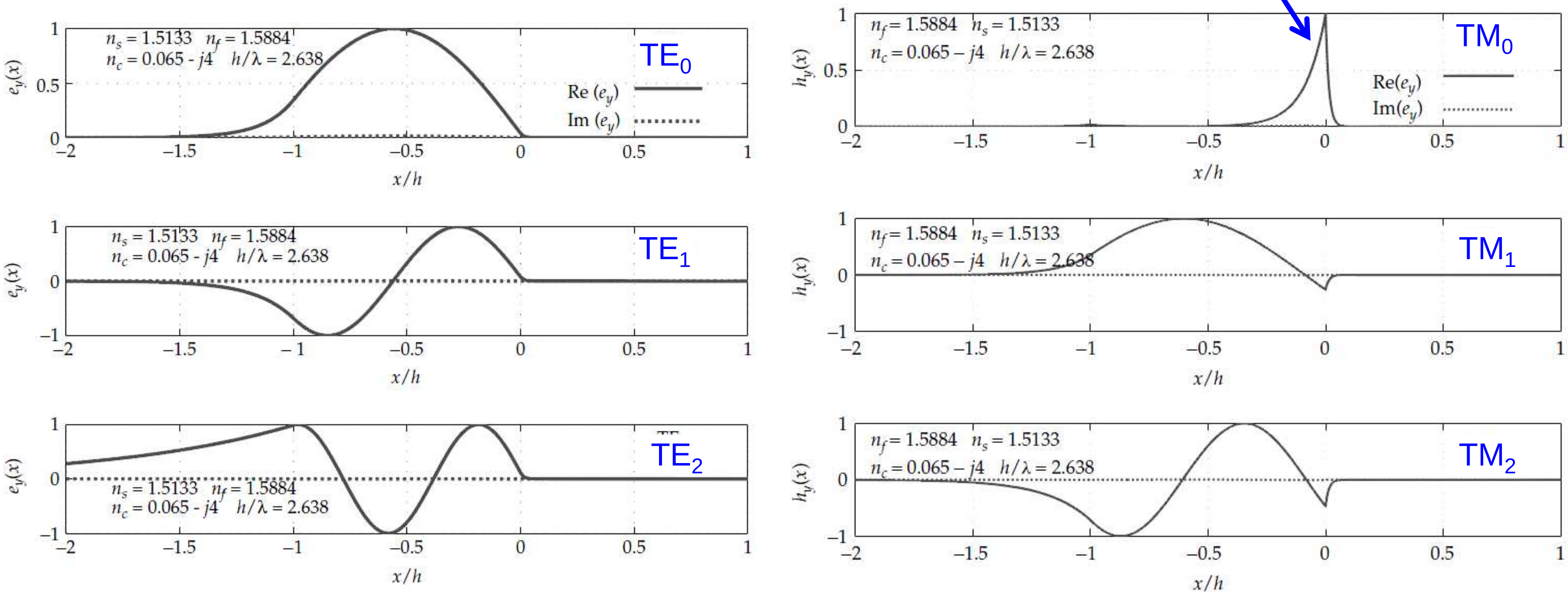
- Propagation constant is nearly independent of frequency
- High attenuation, independent of frequency

Chen, Guided Wave Optics

Mode fields of metal-clad slab waveguide

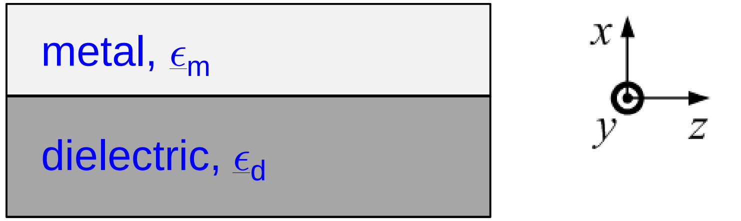
Strong confinement to the metal-dielectric interface!

⇒ Surface plasmon polariton (SPP) mode



First-principle derivation of SPP modes

Ansatz for TM mode confined to the metal-dielectric boundary:



$$\underline{\mathbf{H}}(\mathbf{r},t) = \begin{pmatrix} 0 \\ \underline{H}_y(x) \\ 0 \end{pmatrix} e^{\mathrm{j}(\omega t - \beta z)}$$

where
$$\underline{H}_y(x) = \begin{cases} H_0 e^{-k_{mx}^{(i)} x} & \text{for} \quad x > 0 \\ H_0 e^{k_{dx}^{(i)} x} & \text{for} \quad x < 0 \end{cases} . \qquad k_{mx}^{(i)} = \sqrt{\beta^2 - \epsilon_{rm} k_0^2}, \quad k_{dx}^{(i)} = \sqrt{\beta^2 - \epsilon_{rd} k_0^2},$$

Corresponding electric mode field:

$$\underline{E}_x(x) = \begin{cases} \frac{\beta \underline{H}_y(x)}{\omega \epsilon_0 \underline{\epsilon}_{rm}} & \text{for} \quad x > 0 \\ \frac{\beta \underline{H}_y(x)}{\omega \epsilon_0 \underline{\epsilon}_{rd}} & \text{for} \quad x < 0 \end{cases} ,$$

$$\underline{E}_z(x) = \begin{cases} \mathrm{j} \frac{k_{mx}^{(i)} \underline{H}_y(x)}{\omega \epsilon_0 \underline{\epsilon}_{rm}} & \text{for} \quad x > 0 \\ -\mathrm{j} \frac{k_{dx}^{(i)} \underline{H}_y(x)}{\omega \epsilon_0 \underline{\epsilon}_{rd}} & \text{for} \quad x < 0 \end{cases} ,$$

First-principle derivation of SPP modes

Continuity of E_z at x =0 leads to the dispersion relation of the SPP:

$$\beta = k_0 \sqrt{\frac{\epsilon_{rm}\epsilon_{rd}}{\epsilon_{rm} + \epsilon_{rd}}}$$

The lateral decay constants are given by:

$$k_{mx}^{(i)} = k_0 \sqrt{\frac{-\epsilon_{rm}^2}{\epsilon_{rm} + \epsilon_{rd}}} \qquad k_{dx}^{(i)} = k_0 \sqrt{\frac{-\epsilon_{rd}^2}{\epsilon_{rm} + \epsilon_{rd}}}$$

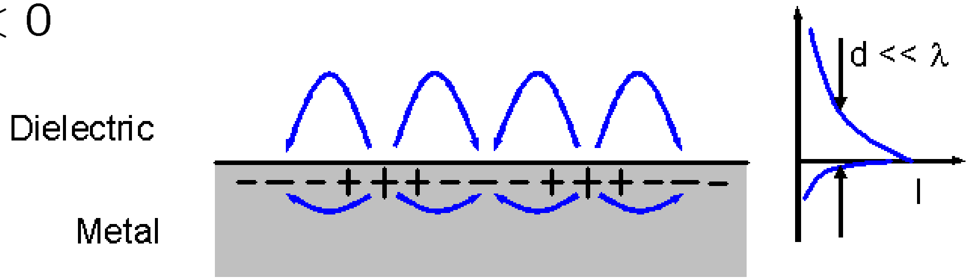
A localized propagating solution requires real β, $k_{mx}^{(i)}$, and $k_{dx}^{(i)}$, i.e.,

$$\epsilon_{rm} < -\epsilon_{rd} \qquad \omega < \frac{\omega_p}{\sqrt{1 + \epsilon_{rd}}}$$

Recall:

$$\underline{E}_x(x) = \begin{cases} \frac{\beta \underline{H}_y(x)}{\omega \epsilon_0 \epsilon_{rm}} & \text{for} \quad x > 0 \\ \frac{\beta \underline{H}_y(x)}{\omega \epsilon_0 \epsilon_{rd}} & \text{for} \quad x < 0 \end{cases},$$

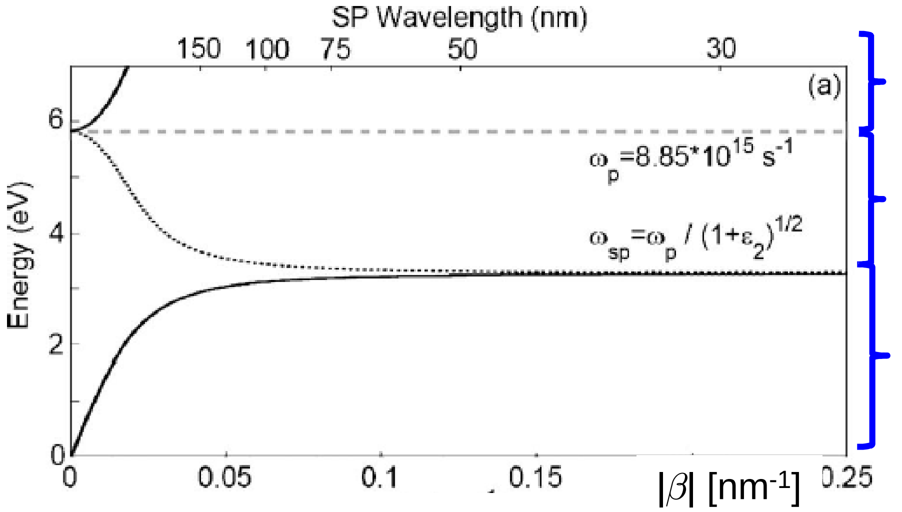
=> E_x must change sign at x = 0:



Institute of Photonics and Quantum Electronics
IPQ

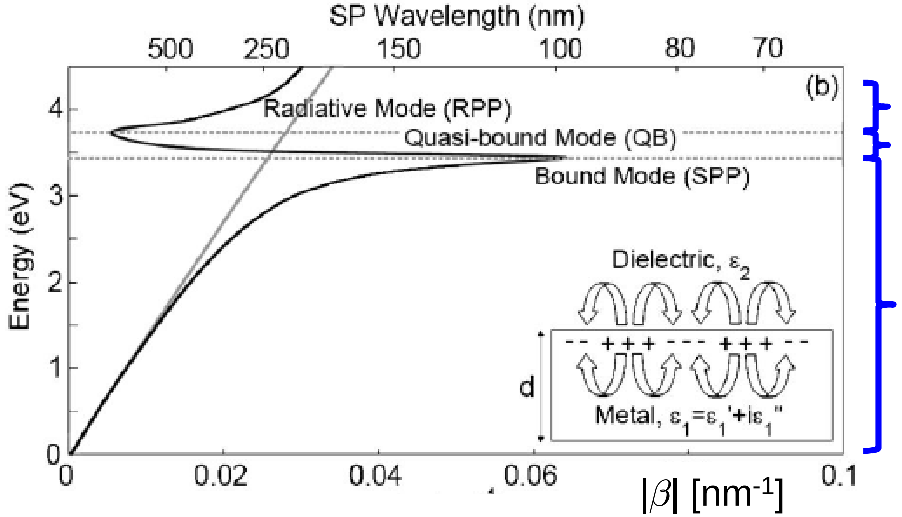
Dispersion relations of surface plasmon polaritons (SPP)

SPP dispersion relation for Ag/SiO_2 interface based on free-electron gas dispersion model:



- Radiative modes
- No propagation, evanescent along the interface
- Bound SPP mode

SPP dispersion relation for Ag/SiO_2 interface based on real material data:



- Radiative modes
- No propagation, evanescent along the interface
- Bound SPP mode

Adapted from Dionne *et al.*, Phys. Rev. B 72, 075406 (2005)

Penetration depths and propagation loss of SPP

Complex dielectric constants of the metal:

$$\underline{\epsilon}_{rm} = \epsilon_{rm} - \mathrm{j}\epsilon_{rm}^{(i)} \quad \text{where usually} \quad \epsilon_{rm}^{(i)} \ll \epsilon_{rm}$$

⇒ Complex propagation constant:

$$\underline{\beta} = \beta - \mathrm{j}\beta_i \quad \text{where (without derivation):} \quad \beta = k_0 \sqrt{\frac{\epsilon_{rm}\epsilon_{rd}}{\epsilon_{rm} + \epsilon_{rd}}}$$

$$\beta_i = k_0 \left(\frac{\epsilon_{rm}\epsilon_{rd}}{\epsilon_{rm} + \epsilon_{rd}} \right)^{3/2} \left(\frac{\epsilon_{rm}^{(i)}}{2\epsilon_{rm}} \right)$$

Lateral penetration depths:

$$\delta_m = 1/k_{mx}^{(i)} \ll \lambda \quad \text{(metal)}$$

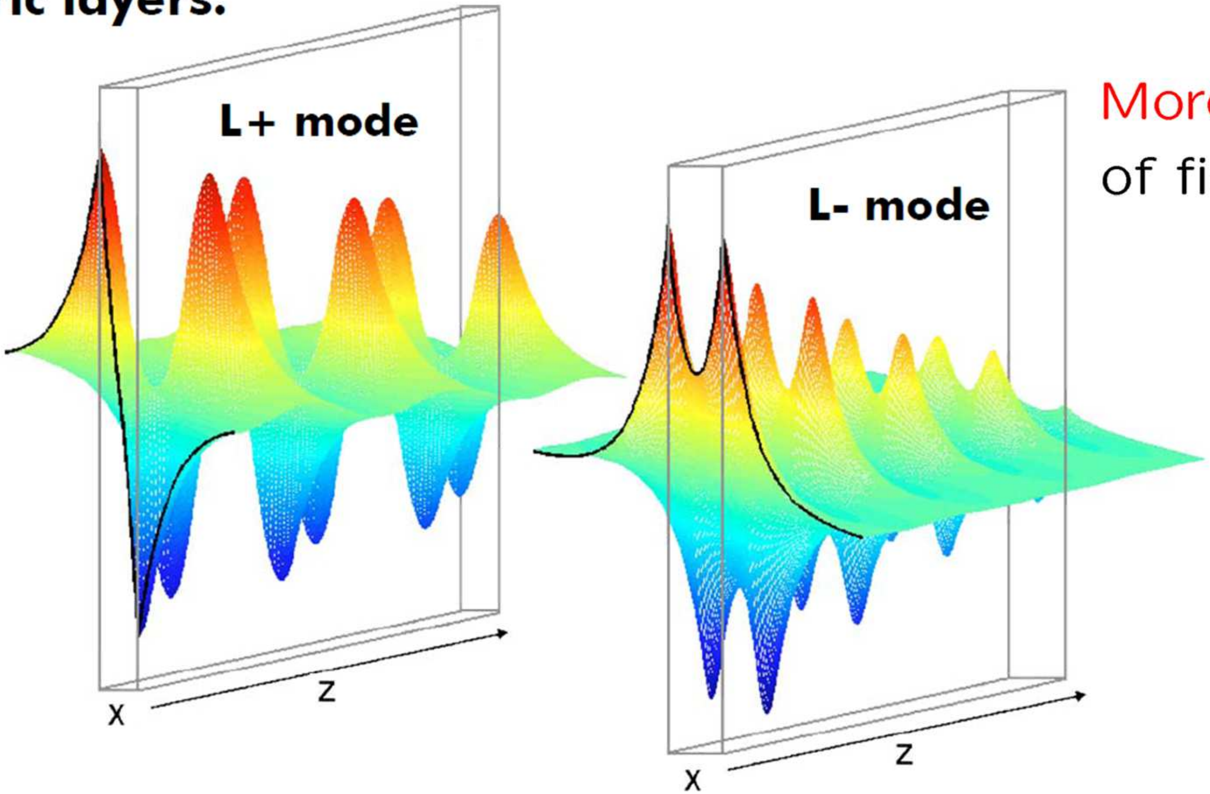
$$\delta_d = 1/k_{dx}^{(i)} \ll \lambda \quad \text{(dielectric)}$$

Discussion:

- Lateral penetration depths can be much smaller than the vacuum wavelength
  => Ultra-compact devices
- Surface plasmons can in principle be lossless if $\underline{\epsilon}_{rm}$ is real! SPP propagation loss is only a consequence of "imperfect" material properties of the metal!
- Propagation distances $L_p = (2\beta_i)^{-1}$ are of the order of tens of microns
- Loss reduction by asymmetric coupled surface plasmons propagating along the surfaces of a thin metal film

Long-reach coupled surface plasmons polaritons

Electric field distribution of coupled surface plasmons supported by a thin metal film "sandwiched" between two identical dielectric layers.



More loss, most of field in metal

Anti-symmetric mode

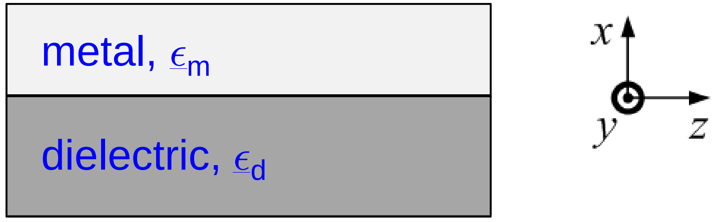
Symmetric mode

Low loss, most of field in dielectric

J. A. Dionne et al., Phys. Rev. B 72, 075405 (2005)

First-principle derivation of SPP modes

Ansatz for TM mode confined to the metal-dielectric boundary:



$$\underline{\mathbf{H}}(\mathbf{r},t) = \begin{pmatrix} 0 \\ \underline{H}_y(x) \\ 0 \end{pmatrix} e^{\mathrm{j}(\omega t - \beta z)}$$

where $$\underline{H}_y(x) = \begin{cases} H_0 e^{-k_{mx}^{(i)} x} & \text{for} \quad x > 0 \\ H_0 e^{k_{dx}^{(i)} x} & \text{for} \quad x < 0 \end{cases}. \qquad k_{mx}^{(i)} = \sqrt{\beta^2 - \epsilon_{rm} k_0^2}, \quad k_{dx}^{(i)} = \sqrt{\beta^2 - \epsilon_{rd} k_0^2},$$

Corresponding electric mode field:

$$\underline{E}_x(x) = \begin{cases} \dfrac{\beta \underline{H}_y(x)}{\omega \epsilon_0 \epsilon_{rm}} & \text{for} \quad x > 0 \\ \dfrac{\beta \underline{H}_y(x)}{\omega \epsilon_0 \epsilon_{rd}} & \text{for} \quad x < 0 \end{cases},$$

$$\underline{E}_z(x) = \begin{cases} \mathrm{j}\dfrac{k_{mx}^{(i)} \underline{H}_y(x)}{\omega \epsilon_0 \epsilon_{rm}} & \text{for} \quad x > 0 \\ -\mathrm{j}\dfrac{k_{dx}^{(i)} \underline{H}_y(x)}{\omega \epsilon_0 \epsilon_{rd}} & \text{for} \quad x < 0 \end{cases},$$

First-principle derivation of SPP modes

Continuity of E_z at x =0 leads to the dispersion relation of the SPP:

$$\beta = k_0 \sqrt{\frac{\epsilon_{rm}\epsilon_{rd}}{\epsilon_{rm} + \epsilon_{rd}}}$$

The lateral decay constants are given by:

$$k_{mx}^{(i)} = k_0 \sqrt{\frac{-\epsilon_{rm}^2}{\epsilon_{rm} + \epsilon_{rd}}} \qquad k_{dx}^{(i)} = k_0 \sqrt{\frac{-\epsilon_{rd}^2}{\epsilon_{rm} + \epsilon_{rd}}}$$

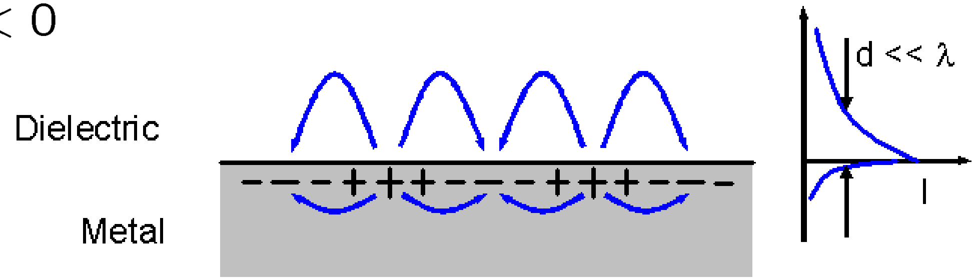
A localized propagating solution requires real β, $k_{mx}^{(i)}$, and $k_{dx}^{(i)}$, i.e.,

$$\epsilon_{rm} < -\epsilon_{rd} \qquad \omega < \frac{\omega_p}{\sqrt{1+\epsilon_{rd}}}$$

Recall:

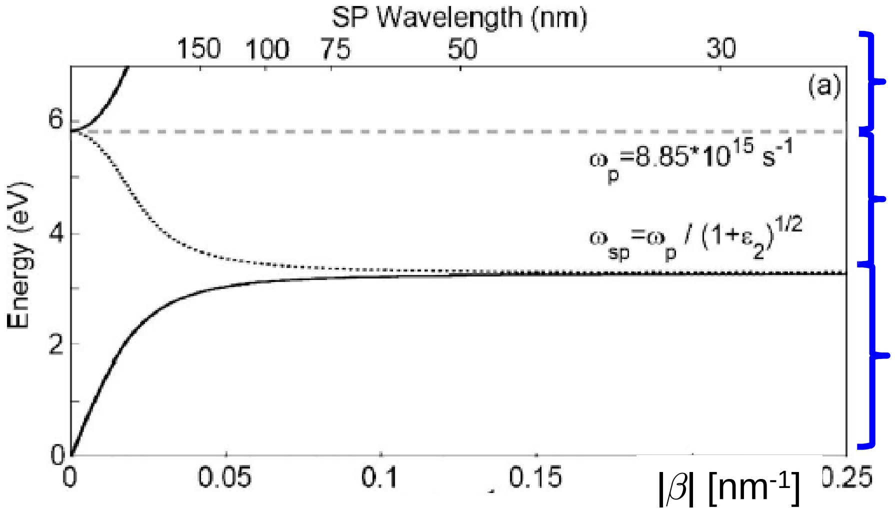
$$\underline{E}_x(x) = \begin{cases} \frac{\beta \underline{H}_y(x)}{\omega \epsilon_0 \epsilon_{rm}} & \text{for} \quad x > 0 \\ \frac{\beta \underline{H}_y(x)}{\omega \epsilon_0 \epsilon_{rd}} & \text{for} \quad x < 0 \end{cases},$$

=> E_x must change sign at x = 0:



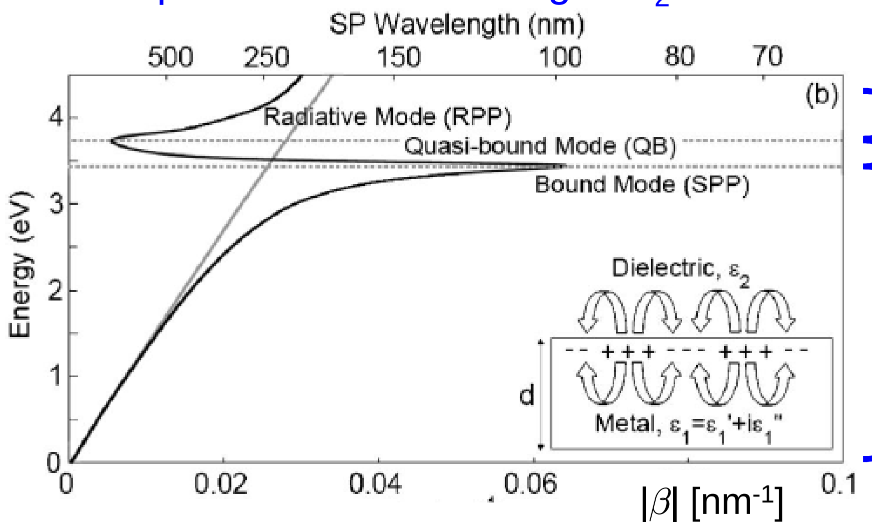
Dispersion relations of surface plasmon polaritons (SPP)

SPP dispersion relation for Ag/SiO_2 interface based on free-electron gas dispersion model:



- Radiative modes
- No propagation, evanescent along the interface
- Bound SPP mode

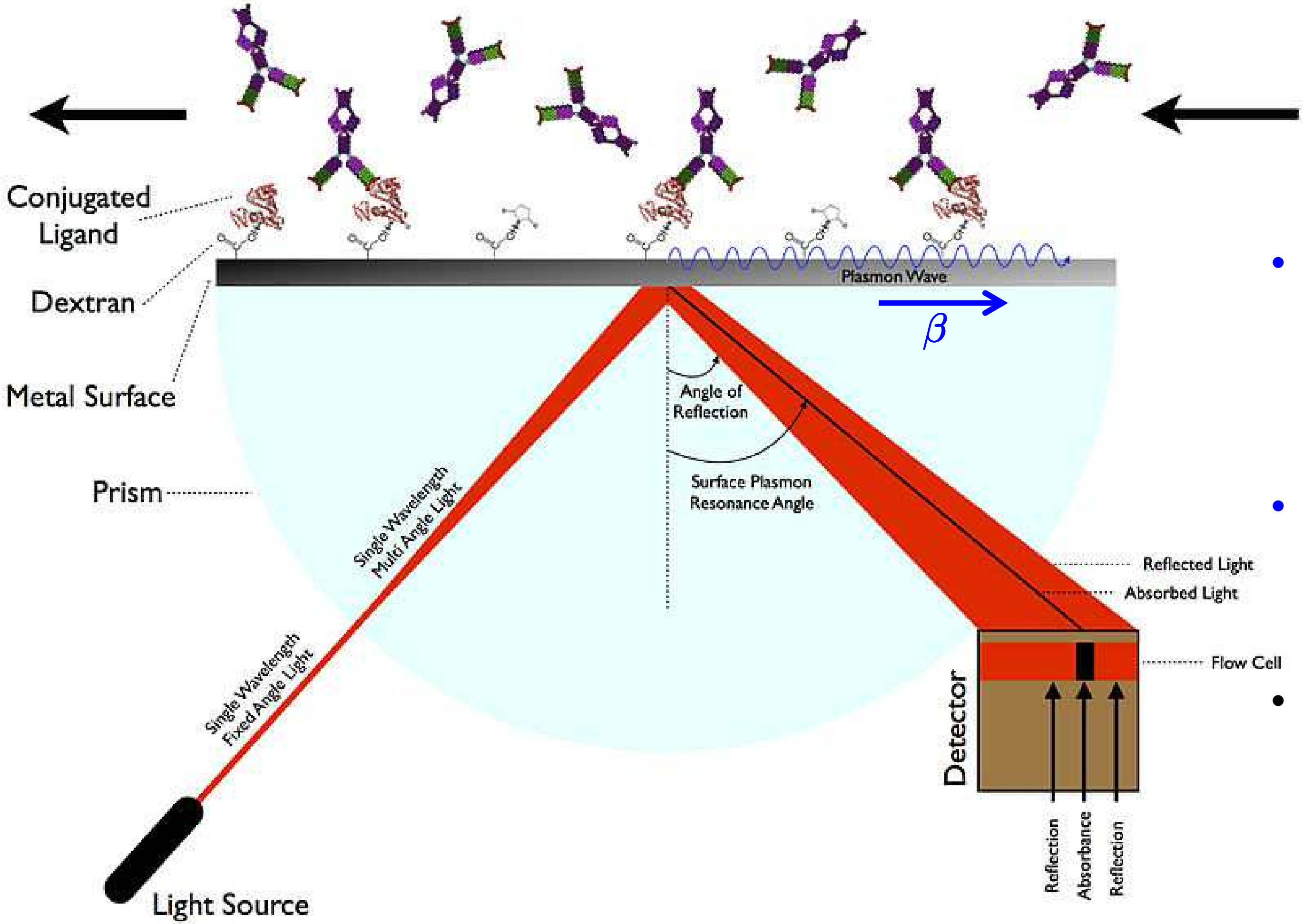
SPP dispersion relation for Ag/SiO_2 interface based on real material data:



- Radiative modes
- No propagation, evanescent along the interface
- Bound SPP mode

Adapted from Dionne *et al.*, Phys. Rev. B 72, 075406 (2005)

Applications of SPP: Bionsensing

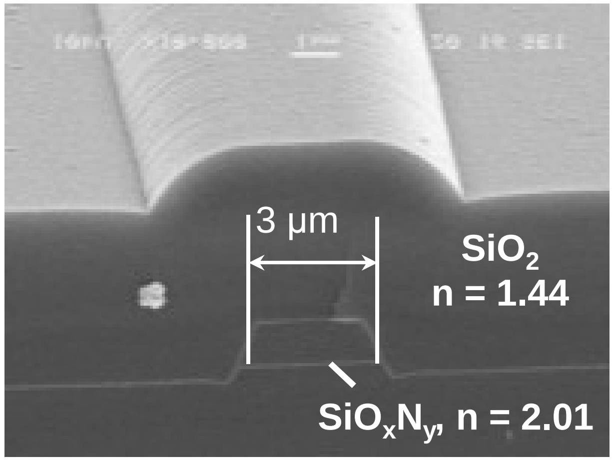


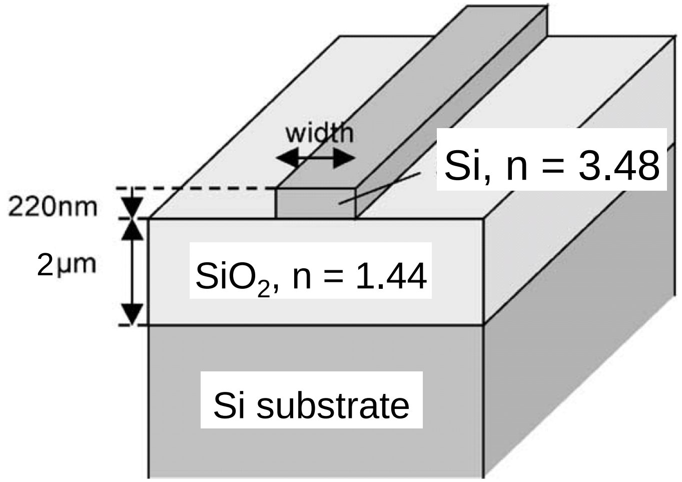
- Excitation of SPP on top surface of the metal for a specific wavenumber β.
  $\Rightarrow$ Dip in the reflected angular spectrum
- Molecular adsorption on the top surface shifts the angle of minimum reflection
- Highly sensitive due to high intensity of SPP mode close to the metal surface

Commercial Product: Biacore, https://www.biacore.com

Figure adapted from www.wikipedia.org

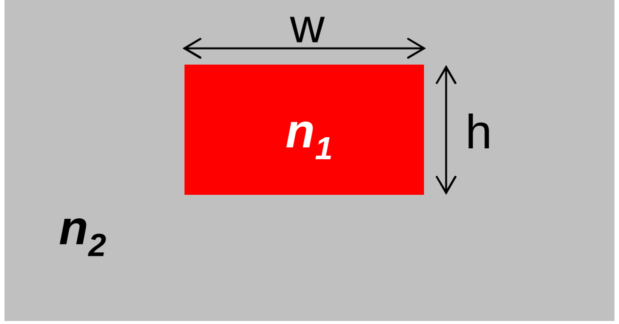
Planar Integrated Waveguides



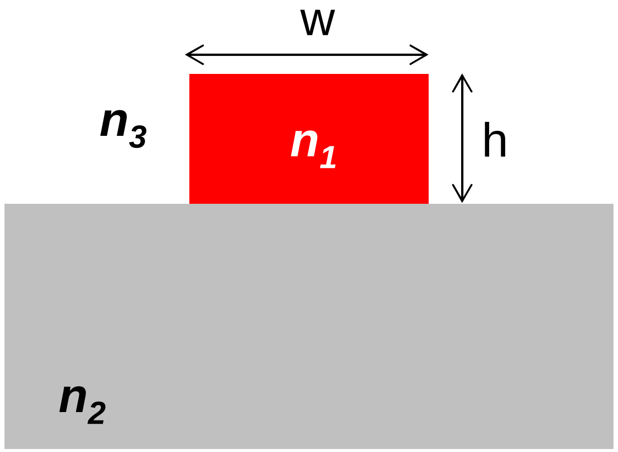


Integrated waveguides

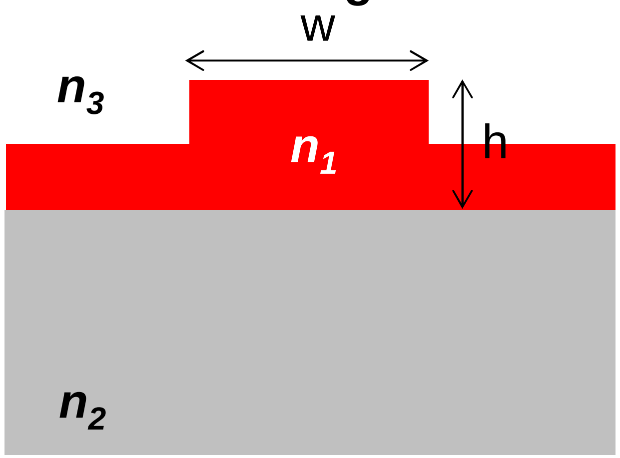
Channel waveguide



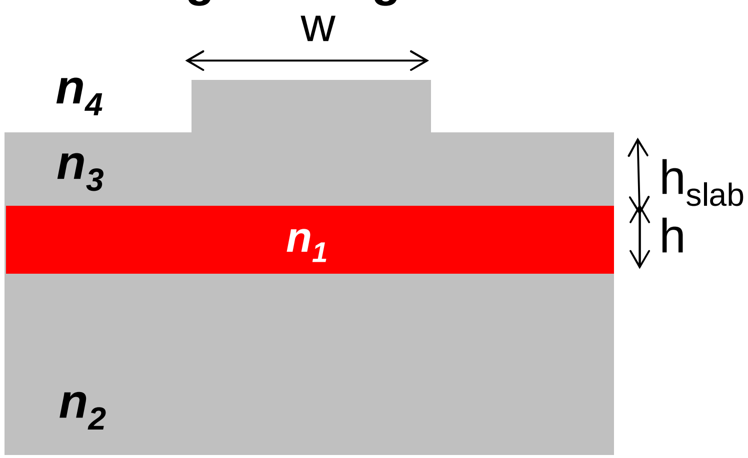
Strip waveguide



Rib waveguide

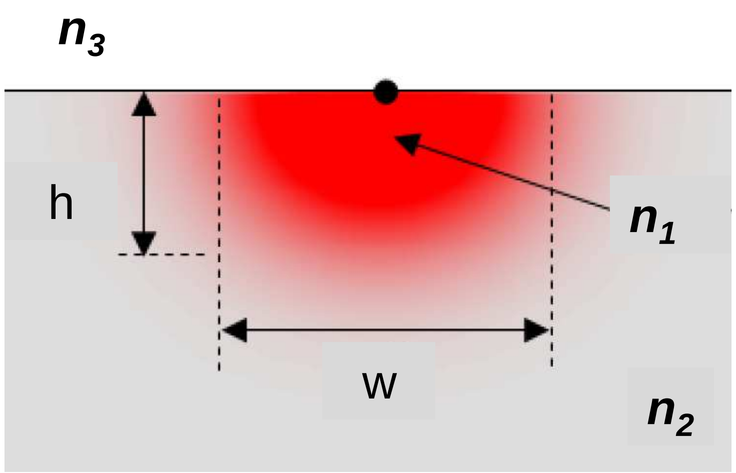


Ridge waveguide

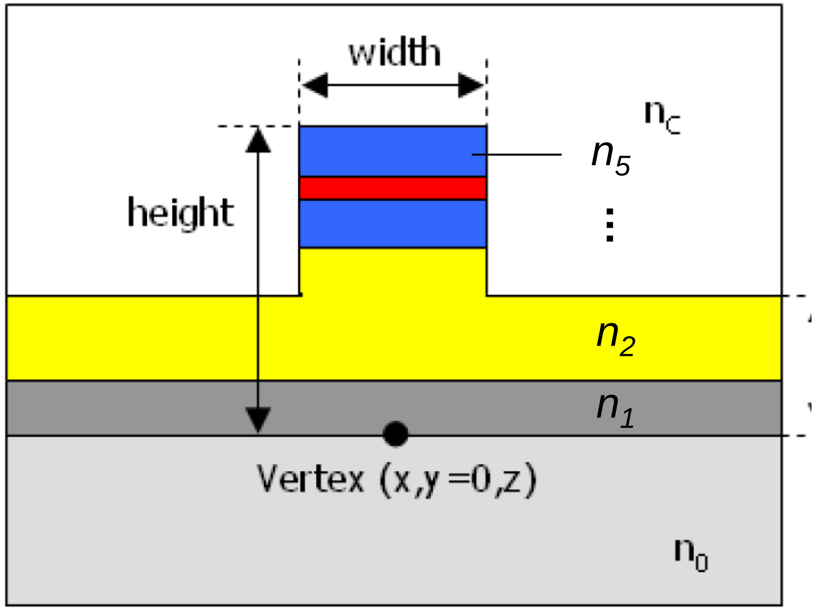


Integrated waveguides

Diffused waveguides



Multilayer waveguides



Guided modes of rectangular channel waveguides: Marcatili method

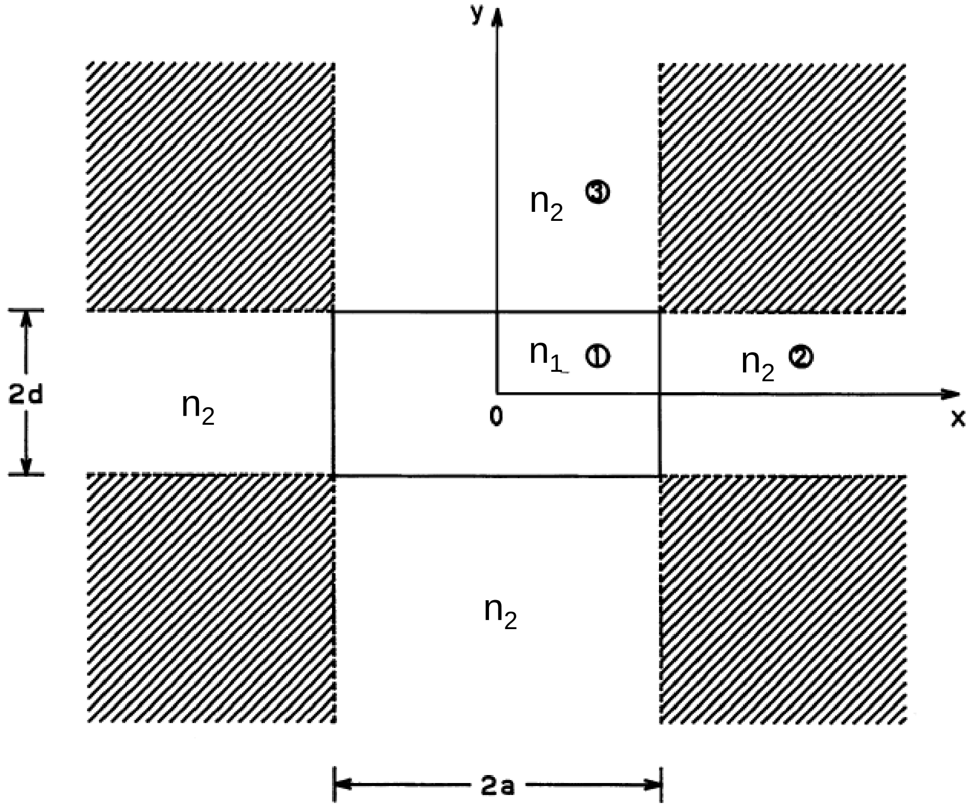
Assumptions:

- Low index contrast: $n_1/n_2 \approx 1$
- Electromagnetic field in the shaded areas can be neglected (field strongly confined to the core)
- Guided fields can be separated in two mode families:

$$\mathcal{H}_x = 0;\ \mathcal{H}_y \text{ and } \mathcal{E}_x \text{ dominate}$$
$$\Rightarrow\ \mathcal{E}_x - \text{mode}$$

$$\mathcal{H}_y = 0;\ \mathcal{H}_x \text{ and } \mathcal{E}_y \text{ dominate}$$
$$\Rightarrow\ \mathcal{E}_y - \text{mode}$$

Marcatili et al.,
Bell Syst. Tech. Journ. 48: 2071 – 2102 (1969)



Solution strategy:

- Start from Maxwell's equations for E_x (E_y) modes and express all field components by H_y (H_x)
- Use mode ansatz for H_y (H_x), derive other field components and match boundary conditions at the core-cladding interfaces

Maxwell's equations for E_x-modes

$$\frac{\partial \underline{\mathcal{E}}_z}{\partial y} + \mathrm{j}\,\beta \underline{\mathcal{E}}_y = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_x$$

$$-\frac{\partial \underline{\mathcal{E}}_z}{\partial x} - \mathrm{j}\,\beta \underline{\mathcal{E}}_x = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_y$$

$$\frac{\partial \underline{\mathcal{E}}_y}{\partial x} - \frac{\partial \underline{\mathcal{E}}_x}{\partial y} = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_z$$

E_x-modes: $H_x = 0$

$$\frac{\partial \underline{\mathcal{E}}_z}{\partial y} + \mathrm{j}\,\beta \underline{\mathcal{E}}_y = 0$$

$$-\frac{\partial \underline{\mathcal{E}}_z}{\partial x} - \mathrm{j}\,\beta \underline{\mathcal{E}}_x = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_y$$

$$\frac{\partial \underline{\mathcal{E}}_y}{\partial x} - \frac{\partial \underline{\mathcal{E}}_x}{\partial y} = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_z$$

$$\frac{\partial \underline{\mathcal{H}}_z}{\partial y} + \mathrm{j}\,\beta \underline{\mathcal{H}}_y = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_x$$

$$-\frac{\partial \underline{\mathcal{H}}_z}{\partial x} - \mathrm{j}\,\beta \underline{\mathcal{H}}_x = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_y$$

$$\frac{\partial \underline{\mathcal{H}}_y}{\partial x} - \frac{\partial \underline{\mathcal{H}}_x}{\partial y} = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_z$$

$\Rightarrow$

$$\frac{\partial \underline{\mathcal{H}}_z}{\partial y} + \mathrm{j}\,\beta \underline{\mathcal{H}}_y = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_x$$

$$-\frac{\partial \underline{\mathcal{H}}_z}{\partial x} = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_y$$

$$\frac{\partial \underline{\mathcal{H}}_y}{\partial x} = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_z$$

$$\frac{\partial \underline{\mathcal{H}}_x}{\partial x} + \frac{\partial \underline{\mathcal{H}}_y}{\partial y} - \mathrm{j}\,\beta \underline{\mathcal{H}}_z = 0$$

$\Rightarrow$

$$\frac{\partial \underline{\mathcal{H}}_y}{\partial y} - \mathrm{j}\,\beta \underline{\mathcal{H}}_z = 0$$

Field components

E_x-modes:

Express all field components by H_y

$$H_x = 0$$

$$E_x = \frac{\omega\mu_0}{\beta} H_y + \frac{1}{\omega\varepsilon_0 n^2 \beta} \frac{\partial^2 H_y}{\partial x^2}$$

$$E_y = \frac{1}{\omega\varepsilon_0 n^2 \beta} \frac{\partial^2 H_y}{\partial x \partial y}$$

$$E_z = \frac{-j}{\omega\varepsilon_0 n^2} \frac{\partial H_y}{\partial x}$$

$$H_z = \frac{-j}{\beta} \frac{\partial H_y}{\partial y}.$$

E_y-modes:

Express all field components by H_x

$$H_y = 0$$

$$E_x = -\frac{1}{\omega\varepsilon_0 n^2 \beta} \frac{\partial^2 H_x}{\partial x \partial y}$$

$$E_y = -\frac{\omega\mu_0}{\beta} H_x - \frac{1}{\omega\varepsilon_0 n^2 \beta} \frac{\partial^2 H_x}{\partial y^2}$$

$$E_z = \frac{j}{\omega\varepsilon_0 n^2} \frac{\partial H_x}{\partial y}$$

$$H_z = \frac{-j}{\beta} \frac{\partial H_x}{\partial x}.$$

Wave equation for dominant components

E_x-modes:

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + (k^2 n^2 - \beta^2) H_y = 0,$$

E_y-modes:

$$\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + (k^2 n^2 - \beta^2) H_x = 0,$$

… within homogeneous core and cladding regions

Mode-field ansatz for E_x-modes:

Symmetry => Consider first quadrant only

$$\underline{\mathcal{H}}_y(x,y) = \begin{cases} A \cos(k_{1x}x - \Phi_x) \cos\left(k_{1y}y - \Phi_y\right) & \text{in region 1} \\ A \cos(k_{1x}a - \Phi_x) \exp\left(-k_{2x}^{(i)}(x-a)\right) \cos\left(k_{1y}y - \Phi_y\right) & \text{in region 2} \\ A \cos(k_{1x}x - \Phi_x) \cos\left(k_{1y}d - \Phi_y\right) \exp\left(-k_{3y}^{(i)}(y-d)\right) & \text{in region 3} \end{cases}$$

$$\begin{aligned} -k_{1x}^2 - k_{1y}^2 - \beta^2 + n_1^2 k_0^2 &= 0 \\ k_{2x}^{(i)\,2} - k_{1y}^2 - \beta^2 + n_2^2 k_0^2 &= 0 \\ -k_{1x}^2 + k_{3y}^{(i)\,2} - \beta^2 + n_2^2 k_0^2 &= 0 \end{aligned}$$

$$\Phi_x = (p-1)\,\frac{\pi}{2} \qquad p = 1, 2\ldots$$

$$\Phi_y = (q-1)\,\frac{\pi}{2} \qquad q = 1, 2\ldots$$

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Relative magnitudes of field components for E_x-modes

Eliminate β :

$$k_{1x}^2 + k_{2x}^{(i)\,2} = \left(n_1^2 - n_2^2\right) k_0^2$$
$$k_{1y}^2 + k_{3x}^{(i)\,2} = \left(n_1^2 - n_2^2\right) k_0^2$$

For low index contrast: $n_1^2 - n_2^2 \ll n_1^2$

$$\Rightarrow k_{1x} \ll n_1 k_0, \qquad k_{1y} \ll n_1 k_0, \qquad \beta \approx n_1 k_0$$

Magnitudes of field components:

$$|\underline{\mathcal{E}}_x| \approx \left|\frac{\omega\mu_0}{\beta}\right| |\underline{\mathcal{H}}_y| \qquad |\underline{\mathcal{E}}_z| \approx \left|\frac{k_{1x}\beta}{n_1^2 k_0^2}\right| |\underline{\mathcal{H}}_y| \approx O\left(\delta\right)\,|\underline{\mathcal{E}}_x|$$

$$|\underline{\mathcal{E}}_y| \approx \left|\frac{k_{1x}^2}{n_1^2 k_0^2}\right| |\underline{\mathcal{E}}_x| \approx O\left(\delta^2\right)\,|\underline{\mathcal{E}}_x| \qquad |\underline{\mathcal{H}}_z| \approx \left|\frac{k_{1y}}{\beta}\right| |\underline{\mathcal{H}}_y| \approx O\left(\delta\right)\,|\underline{\mathcal{H}}_y|$$

where $\delta \approx \dfrac{k_{1x}}{n_1 k_0} \approx \dfrac{k_{1y}}{n_1 k_0} \ll 1.$

Conclusion: Match boundary conditions for E_z and H_z, ignore E_y!

Longitudinal field components of E_x-modes

$$\underline{\mathcal{H}}_y(x,y) = \begin{cases} A\cos(k_{1x}x-\Phi_x)\cos\left(k_{1y}y-\Phi_y\right) & \text{in region 1} \\ A\cos(k_{1x}a-\Phi_x)\exp\left(-k_{2x}^{(i)}(x-a)\right)\cos\left(k_{1y}y-\Phi_y\right) & \text{in region 2} \\ A\cos(k_{1x}x-\Phi_x)\cos\left(k_{1y}d-\Phi_y\right)\exp\left(-k_{3y}^{(i)}(y-d)\right) & \text{in region 3} \end{cases}$$

Calculate longitudinal field components: $\underline{\mathcal{E}}_z = \frac{1}{\mathrm{j}\omega\epsilon_0 n^2}\frac{\partial \underline{\mathcal{H}}_y}{\partial x}$ $\underline{\mathcal{H}}_z = \frac{1}{\mathrm{j}\beta}\frac{\partial \underline{\mathcal{H}}_y}{\partial y}$

$$\underline{\mathcal{E}}_z(x,y) = \frac{A}{\mathrm{j}\,\omega\epsilon_0}\begin{cases} -\frac{k_{1x}}{n_1^2}\sin(k_{1x}x-\Phi_x)\cos\left(k_{1y}y-\Phi_y\right) & \text{in region 1} \\ -\frac{k_{2x}^{(i)}}{n_2^2}\cos(k_{1x}a-\Phi_x)\exp\left(-k_{2x}^{(i)}(x-a)\right)\cos\left(k_{1y}y-\Phi_y\right) & \text{in region 2} \\ -\frac{k_{1x}}{n_2^2}\sin(k_{1x}x-\Phi_x)\cos\left(k_{1y}d-\Phi_y\right)\exp\left(-k_{3y}^{(i)}(y-d)\right) & \text{in region 3} \end{cases}$$

$$\underline{\mathcal{H}}_z(x,y) = \frac{A}{\mathrm{j}\,\beta}\begin{cases} -k_{1y}\cos(k_{1x}x-\Phi_x)\sin\left(k_{1y}y-\Phi_y\right) & \text{in region 1} \\ -k_{1y}\cos(k_{1x}a-\Phi_x)\exp\left(-k_{2x}^{(i)}(x-a)\right)\sin\left(k_{1y}y-\Phi_y\right) & \text{in region 2}\ . \\ -k_{3y}^{(i)}\cos(k_{1x}x-\Phi_x)\cos\left(k_{1y}d-\Phi_y\right)\exp\left(-k_{3y}^{(i)}(y-d)\right) & \text{in region 3} \end{cases}$$

Dispersion relations for E_x modes

Dispersion equation for $\mathcal{E}_x^{(p,q)}$-modes:

$$k_{1x}a = (p-1)\frac{\pi}{2} + \arctan\left(\frac{n_1^2\, k_{2x}^{(i)}}{n_2^2\, k_{1x}}\right)$$

$$k_{1y}d = (q-1)\frac{\pi}{2} + \arctan\left(\frac{k_{3y}^{(i)}}{k_{1y}}\right)$$

Numerical solution:

- **Insert**

$$k_{2x}^{(i)\,2} = k_0^2\left(n_1^2 - n_2^2\right) - k_{1x}^2$$

$$k_{3y}^{(i)\,2} = k_0^2\left(n_1^2 - n_2^2\right) - k_{1y}^2$$

into dispersion equations and solve for k_{1x} and k_{1y}

- **The propagation constant is obtained by:**

$$\beta^2 = k_0^2 n_1^2 - k_{1x}^2 - k_{1y}^2$$

E_y modes

Karlsruhe Institute of Technology

Similar derivation as for E_x-modes …

Dispersion equation for $\mathcal{E}_y^{(p,q)}$-modes:

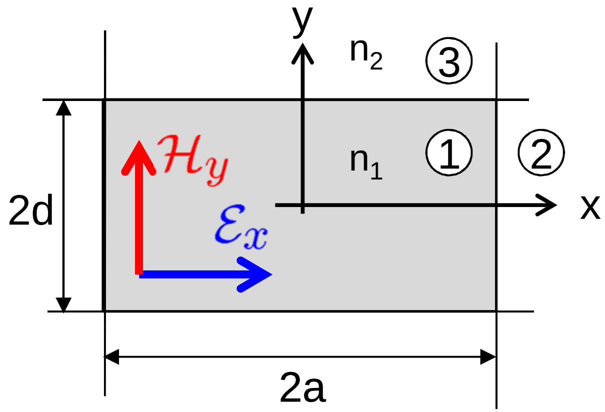
$$k_{1x}a = (p-1)\frac{\pi}{2} + \arctan\left(\frac{k_{2x}^{(i)}}{k_{1x}}\right)$$

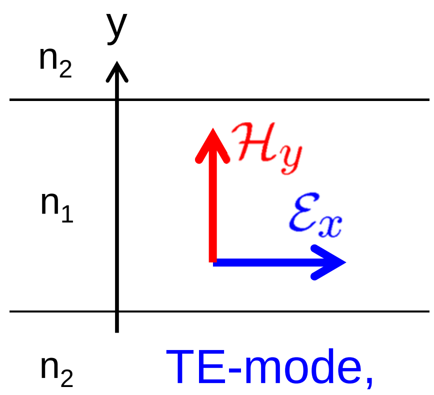
$$k_{1y}d = (q-1)\frac{\pi}{2} + \arctan\left(\frac{n_1^2\, k_{3y}^{(i)}}{n_2^2\, k_{1y}}\right)$$

Dispersion equation of conventional slab waveguide

Slab waveguide interpretation of Marcatili method: E_x

Dominant components of $\mathcal{E}_x^{(p,q)}$-modes: $\mathcal{E}_x$ and $\mathcal{H}_y$



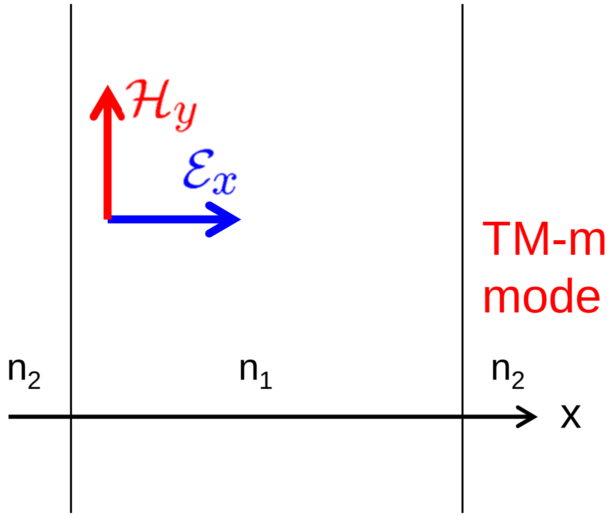


Equivalent horizontal slab waveguide

TE-mode,
mode index $m = q - 1$

$$k_{1y} d = (q-1)\frac{\pi}{2} + \arctan\left(\frac{k_{3y}^{(i)}}{k_{1y}}\right)$$

Equivalent to slab waveguide eigenvalue equations derived earlier!



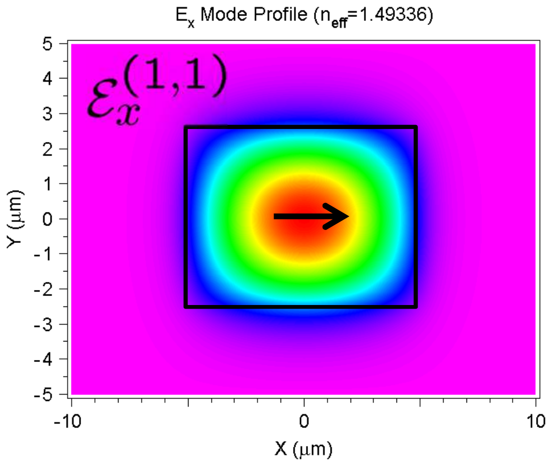
TM-mode,
mode index $m = p - 1$

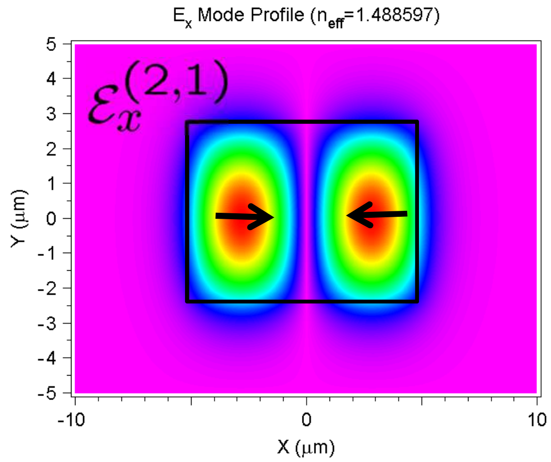
$$k_{1x} a = (p-1)\frac{\pi}{2} + \arctan\left(\frac{n_1^2\, k_{2x}^{(i)}}{n_2^2\, k_{1x}}\right)$$

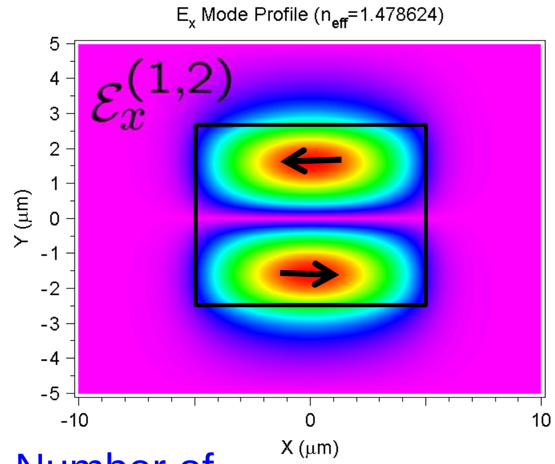
Equivalent vertical slab waveguide

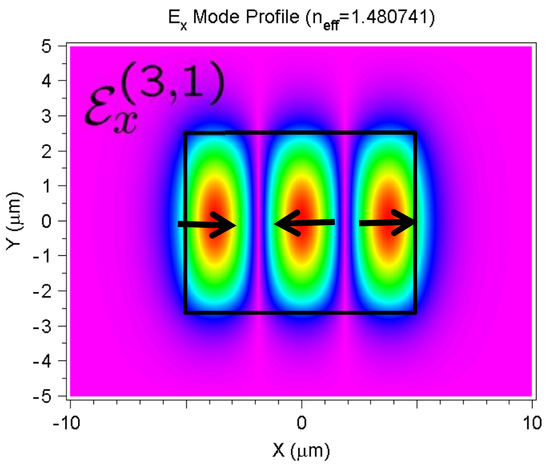
E_x-modes of a channel waveguide

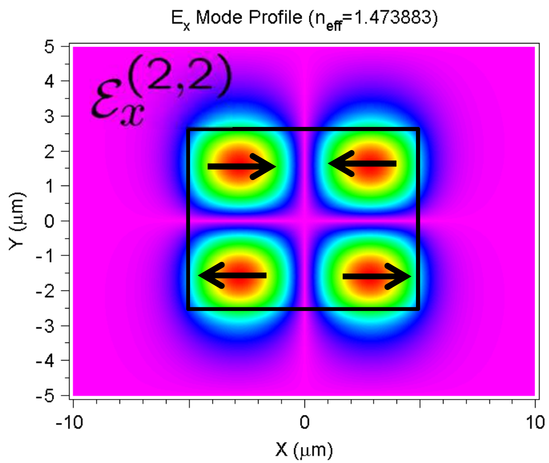
n_1 = 1.5; n_2 = 1.45, width: 2a = 20 µm, height: 2d =10 µm, λ = 1.55 µm











Number of intensity maxima in x-direction

Number of intensity maxima in y-direction

$\mathcal{E}_x^{(p,q)}$-modes = (quasi-)TE mode

Dominant E-field component

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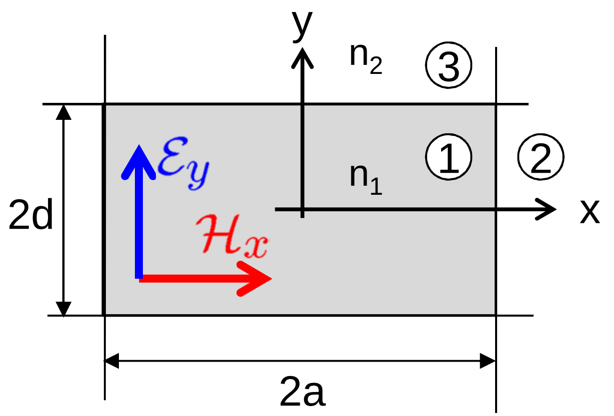
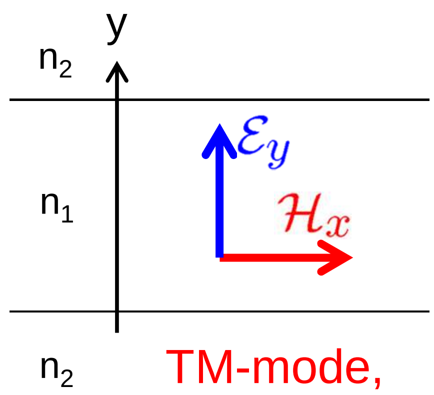
Slab waveguide interpretation of Marcatili method: E_y

Dominant components of $\mathcal{E}_y^{(p,q)}$-modes: $\mathcal{E}_y$ and $\mathcal{H}_x$



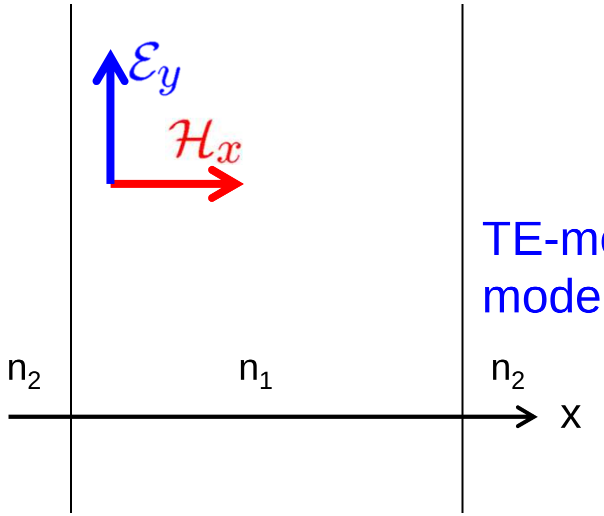


equivalent horizontal slab waveguide

TM-mode,
mode index $m = q - 1$

$$k_{1y}d = (q-1)\frac{\pi}{2} + \arctan\left(\frac{n_1^2\,k_{3y}^{(i)}}{n_2^2\,k_{1y}}\right)$$

Equivalent to slab waveguide eigenvalue equations derived earlier!



TE-mode,
mode index $m = p - 1$

$$k_{1x}a = (p-1)\frac{\pi}{2} + \arctan\left(\frac{k_{2x}^{(i)}}{k_{1x}}\right)$$

equivalent vertical slab waveguide

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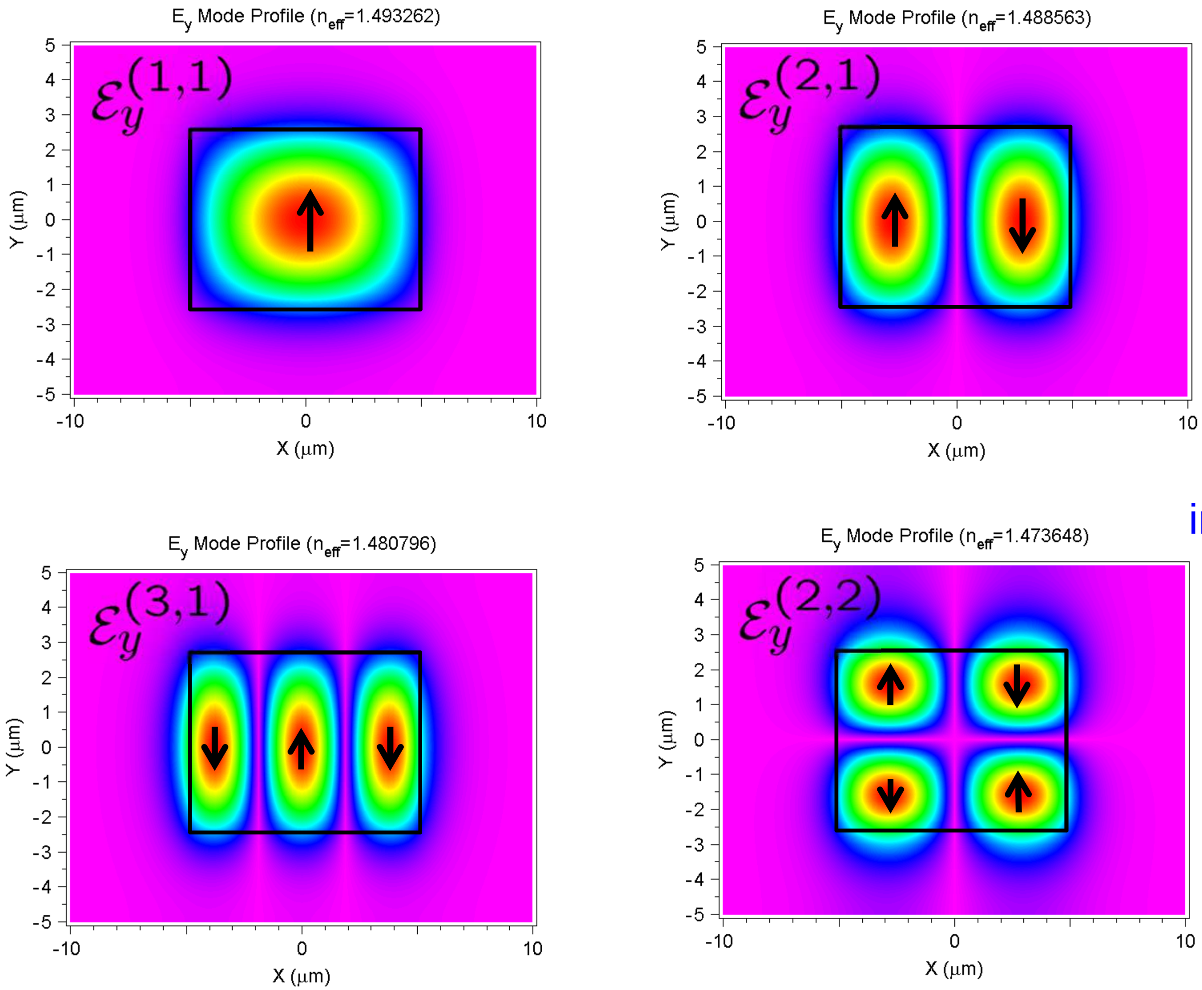
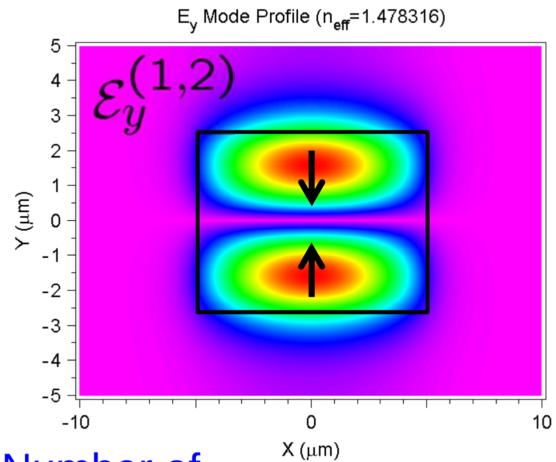
E_y-modes of a channel waveguide

n_1 = 1.5; n_2 = 1.45, width: 2a = 20 µm, height: 2d =10 µm, λ = 1.55 µm





Number of intensity maxima in x-direction

Number of intensity maxima in y-direction

$\mathcal{E}_y^{(p,q)}$-modes

= (quasi-)TM mode

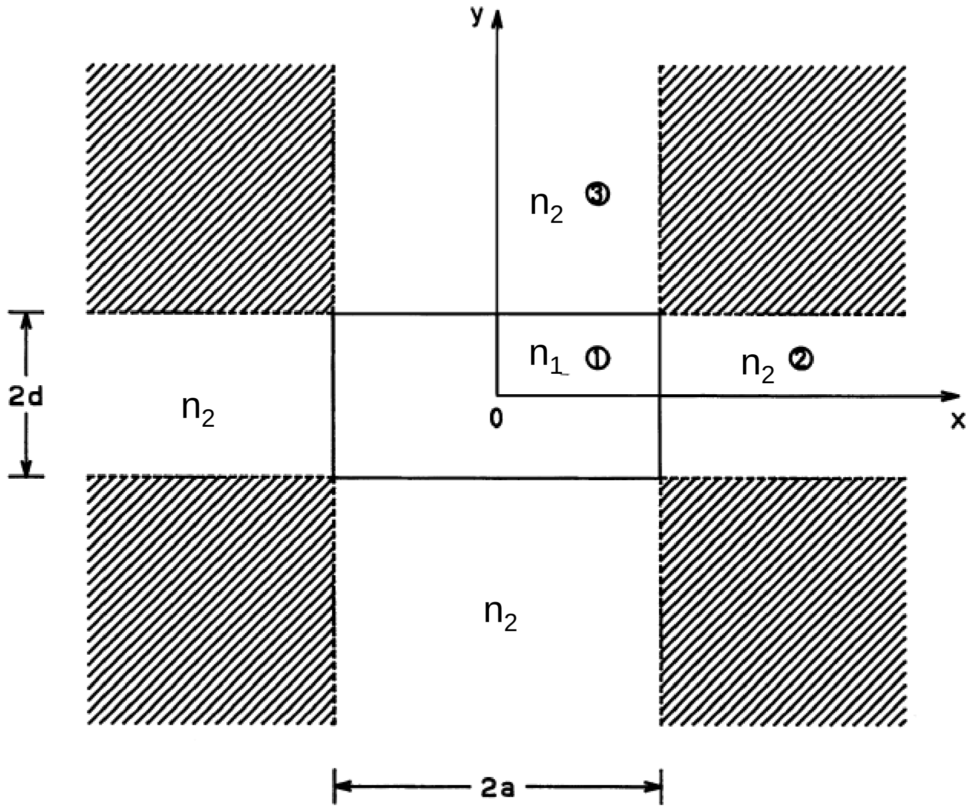
Dominant E-field component

Assumptions:

- Low index contrast: $n_1/n_2 \approx 1$
- Electromagnetic field in the shaded areas can be neglected (field strongly confined to the core)
- Guided fields can be separated in two modes:

$$\underline{\mathcal{H}}_x = 0;\ \underline{\mathcal{H}}_y \text{ and } \underline{\mathcal{E}}_x \text{ dominate}$$
$$\Rightarrow\ \mathcal{E}_x - \text{mode}$$

$$\underline{\mathcal{H}}_y = 0;\ \underline{\mathcal{H}}_x \text{ and } \underline{\mathcal{E}}_y \text{ dominate}$$
$$\Rightarrow\ \mathcal{E}_y - \text{mode}$$

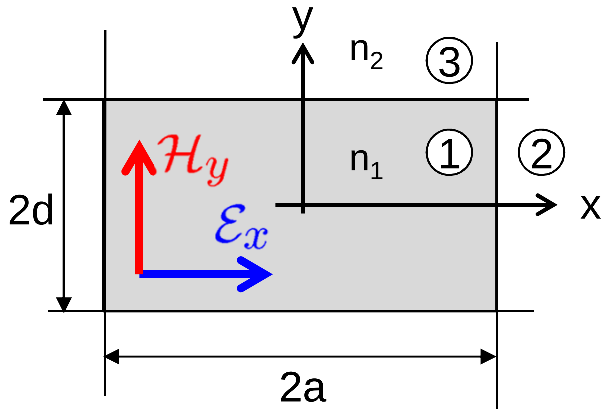


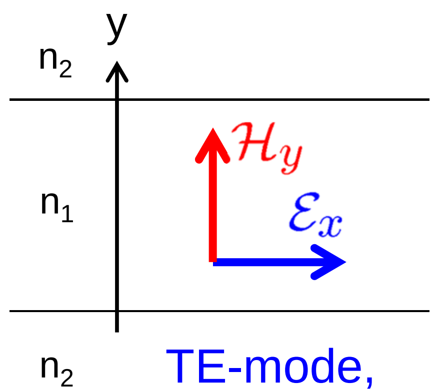
Ansatz for E_x-modes:

$$\underline{\mathcal{H}}_y(x,y) = \begin{cases} A\cos(k_{1x}x - \Phi_x)\cos(k_{1y}y - \Phi_y) & \text{in region 1} \\ A\cos(k_{1x}a - \Phi_x)\exp\left(-k_{2x}^{(i)}(x-a)\right)\cos(k_{1y}y - \Phi_y) & \text{in region 2} \\ A\cos(k_{1x}x - \Phi_x)\cos(k_{1y}d - \Phi_y)\exp\left(-k_{3y}^{(i)}(y-d)\right) & \text{in region 3} \end{cases}$$

Slab waveguide interpretation of Marcatili method: E_x

Dominant components of $\mathcal{E}_x^{(p,q)}$-modes: $\mathcal{E}_x$ and $\mathcal{H}_y$



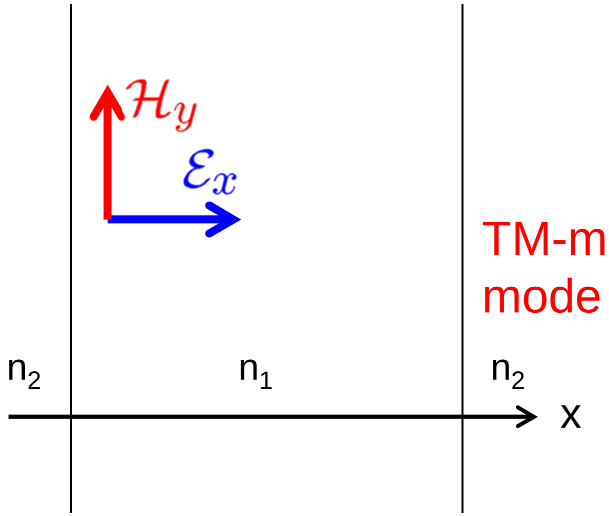


Equivalent horizontal slab waveguide

TE-mode,
mode index $m = q - 1$

$$k_{1y}d = (q-1)\frac{\pi}{2} + \arctan\left(\frac{k_{3y}^{(i)}}{k_{1y}}\right)$$

Equivalent to slab waveguide eigenvalue equations derived earlier!



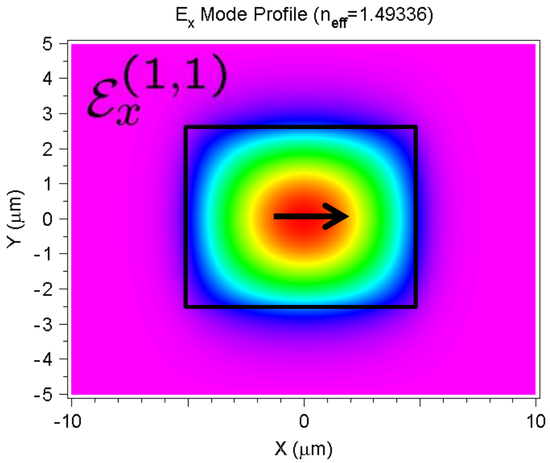
TM-mode,
mode index $m = p - 1$

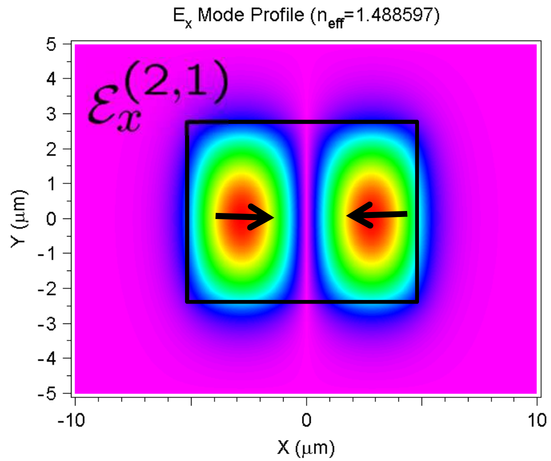
$$k_{1x}a = (p-1)\frac{\pi}{2} + \arctan\left(\frac{n_1^2\, k_{2x}^{(i)}}{n_2^2\, k_{1x}}\right)$$

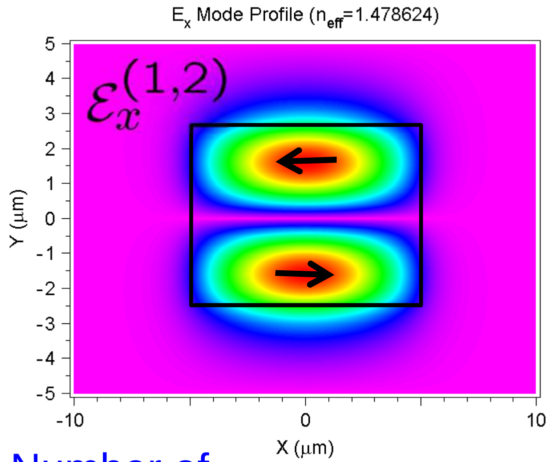
Equivalent vertical slab waveguide

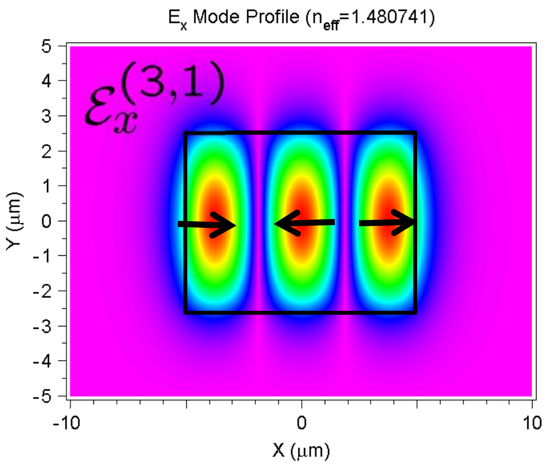
E_x-modes of a channel waveguide

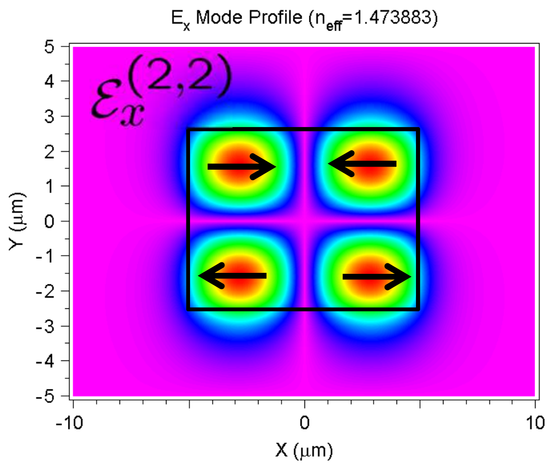
$n_1 = 1.5$; $n_2 = 1.45$, width: $2a = 20\ \mu m$, height: $2d = 10\ \mu m$, $\lambda = 1.55\ \mu m$











Number of intensity maxima in x-direction

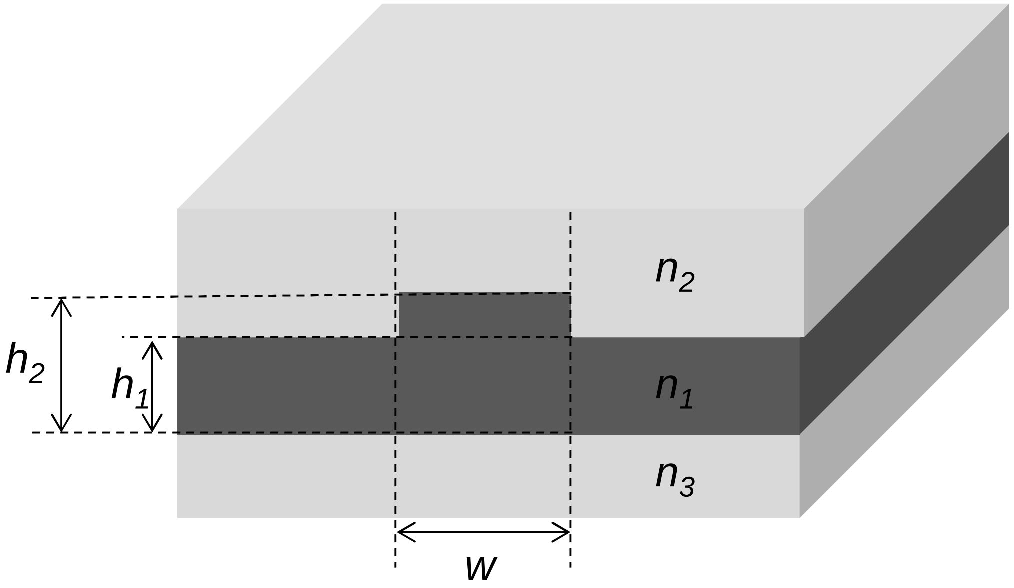
Number of intensity maxima in y-direction

$\mathcal{E}_x^{(p,q)}$-modes

= (quasi-)TE mode

Dominant E-field component

Effective-index method



Assume:

- Low index contrast, $n_1/n_2 \approx n_1/n_3 \approx 1$, i.e. wave equations for weakly inhomogeneous media can be used,

$$\nabla^2 \underline{\mathbf{E}}(\mathbf{r}) + k_0^2 n^2(\mathbf{r})\, \underline{\mathbf{E}}(\mathbf{r}) = 0$$
$$\nabla^2 \underline{\mathbf{H}}(\mathbf{r}) + k_0^2 n^2(\mathbf{r})\, \underline{\mathbf{H}}(\mathbf{r}) = 0$$

- Horizontal waveguide dimensions larger than vertical dimensions, $w >> h_{1,2}$

Effective-index method

Mode field equation for weakly inhomogeneous media:

$$\frac{\partial^2 \underline{\Psi}(x,y)}{\partial x^2} + \frac{\partial^2 \underline{\Psi}(x,y)}{\partial y^2} + \left(k_0^2 n^2(x,y) - \beta^2\right) \underline{\Psi}(x,y) = 0$$

$\underline{\Psi}(x,y)$ can be any component of the mode fields $\underline{\mathcal{E}}(x,y)$ and $\underline{H}(x,y)$.

Assumption: Rapidly varying x-dependence of the electromagnetic fields can be factored out,

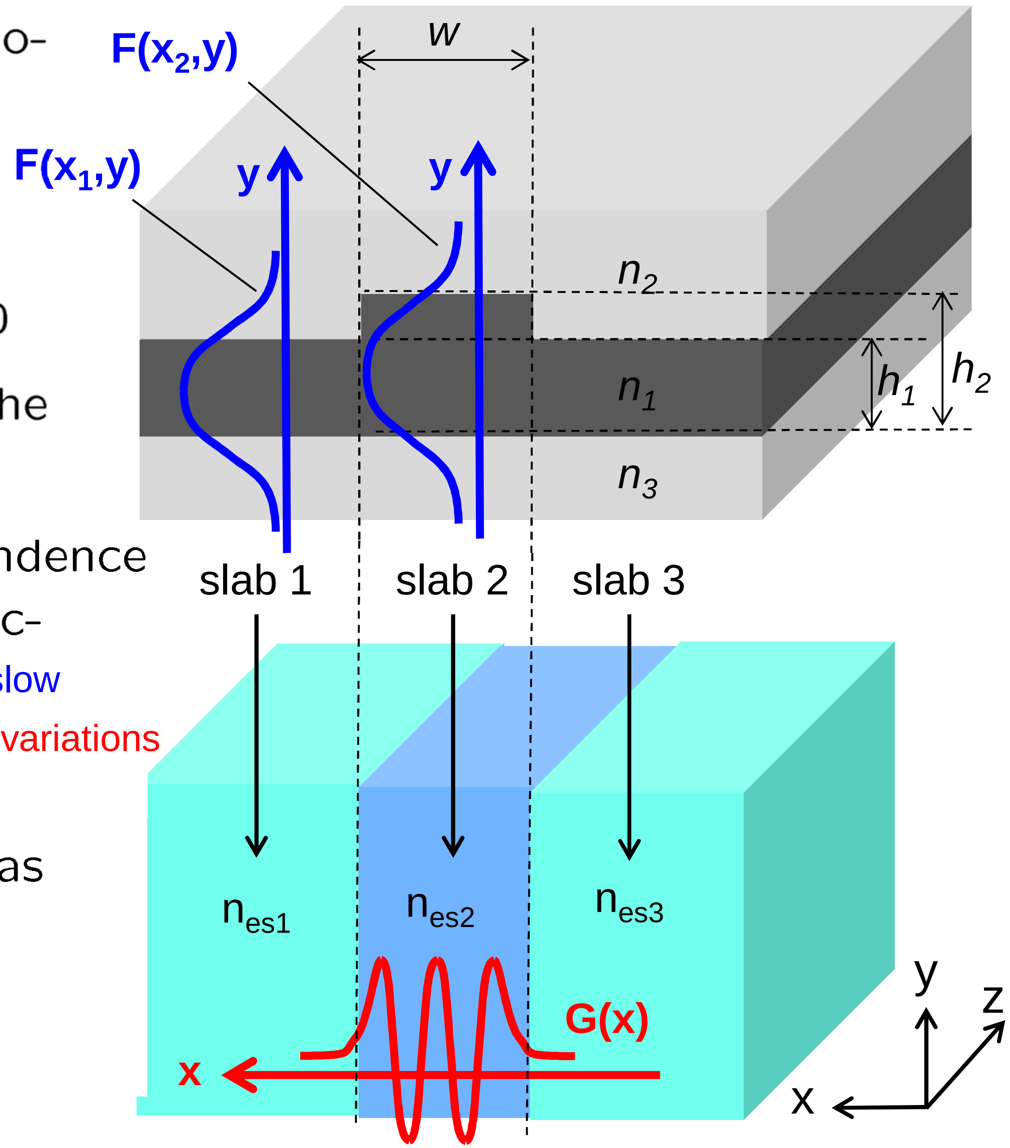
Local y-dependence, slow variations in x

Fast variations in x

$$\underline{\Psi}(x,y) = F(x,y)\, G(x),$$

where $F(x,y)$ is varying only slowly as a function of x,

$$\frac{\partial^2 \underline{\Psi}(x,y)}{\partial x^2} \approx F(x,y) \frac{\partial^2 G(x)}{\partial x^2}.$$



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Effective index method

Wave equation:

$$F(x,y)\,\frac{\partial^2 G(x)}{\partial x^2} + G(x)\,\frac{\partial^2 F(x,y)}{\partial y^2} + \left(k_0^2 n^2(x,y) - \beta^2\right) F(x,y)\,G(x) = 0$$

$$\underbrace{\frac{1}{G(x)}\frac{\partial^2 G(x)}{\partial x^2} - \beta^2}_{\text{purely } x\text{-dependent}} + \underbrace{\frac{1}{F(x,y)}\frac{\partial^2 F(x,y)}{\partial y^2} + k_0^2 n^2(x,y)}_{y-\text{dependence dominates}} = 0$$

Introduce the effective index $n_{es}(x)$ of the horizontal slab waveguide structure as an x-dependent separation variable:

$$\frac{1}{F(x,y)}\frac{\partial^2 F(x,y)}{\partial y^2} + k_0^2 n^2(x,y) = k_0^2 n_{es}^2(x)$$

$$\frac{1}{G(x)}\frac{\partial^2 G(x)}{\partial x^2} - \beta^2 = -k_0^2 n_{es}^2(x)$$

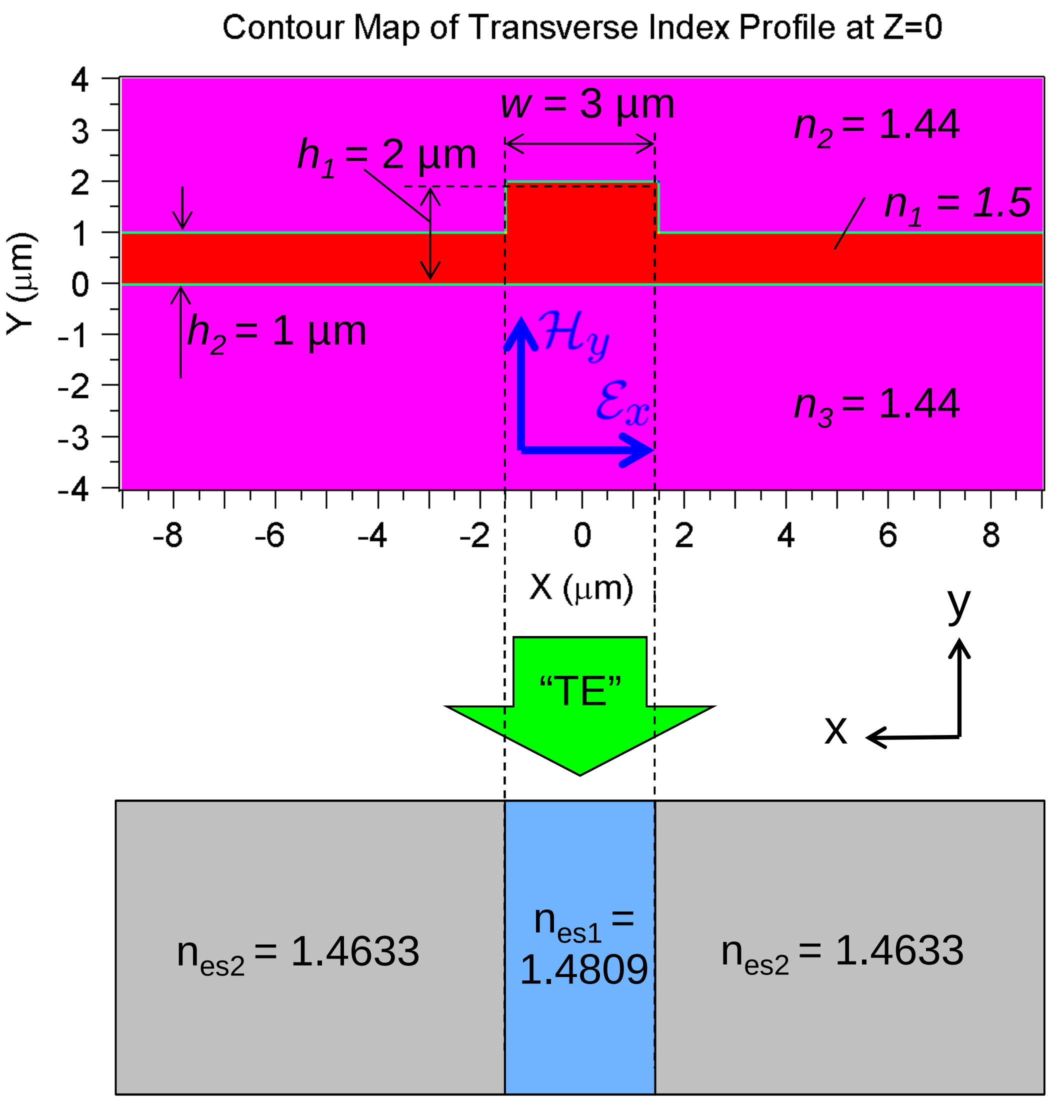
This yields two **related slab waveguide equations**:

$$\frac{\partial^2 F(x,y)}{\partial y^2} + \left(k_0^2 n^2(x,y) - k_0^2 n_{es}^2(x)\right) F(x,y) = 0$$

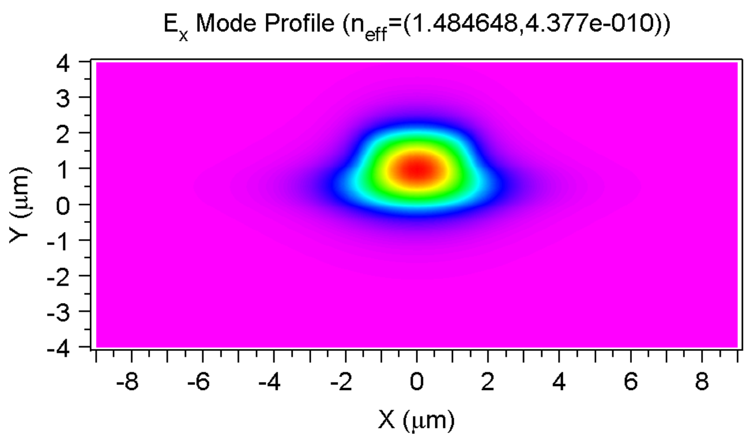
$$\frac{\partial^2 G(x)}{\partial x^2} + \left(k_0^2 n_{es}^2(x) - \beta^2\right) G(x) = 0$$

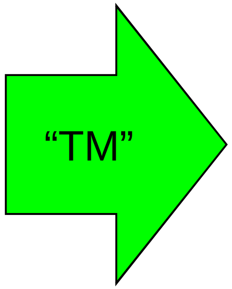
Example: E_x-modes of a rib waveguide

$\lambda = 1.55\ \mu m$



Reference: Numerical solution

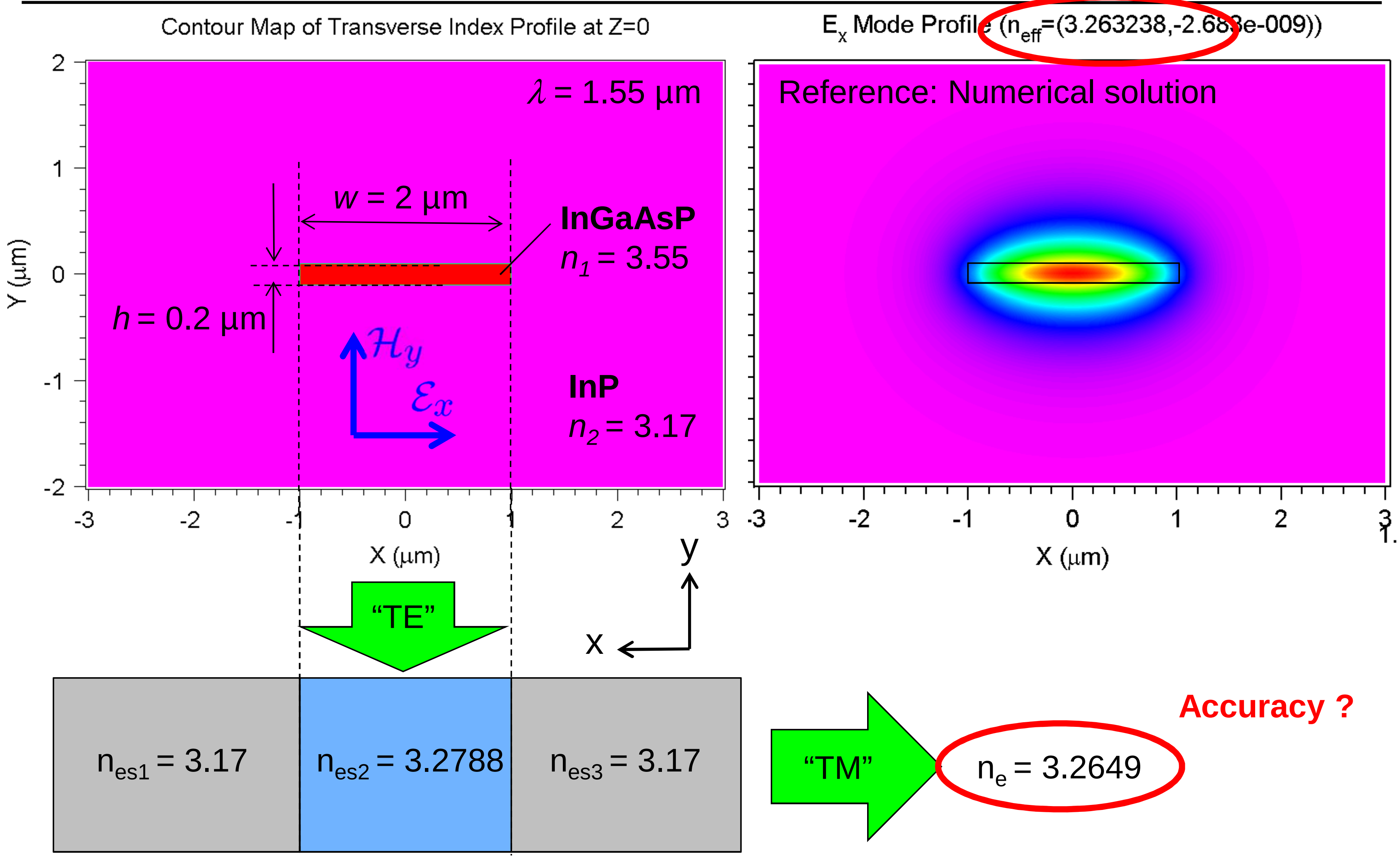




$n_e = 1.4737$

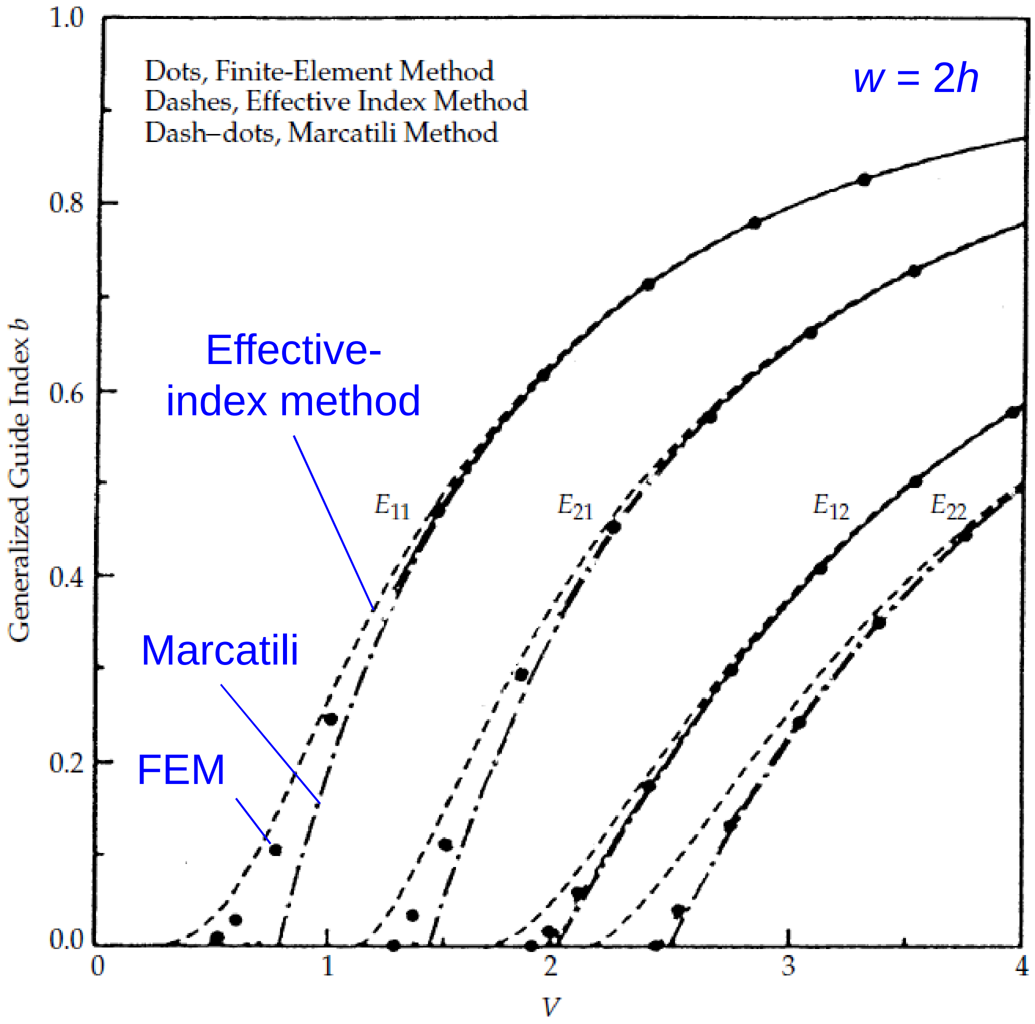
Slab waveguide solver: http://www.computational-photonics.eu/oms.html

Example: E_x-mode of a channel waveguide

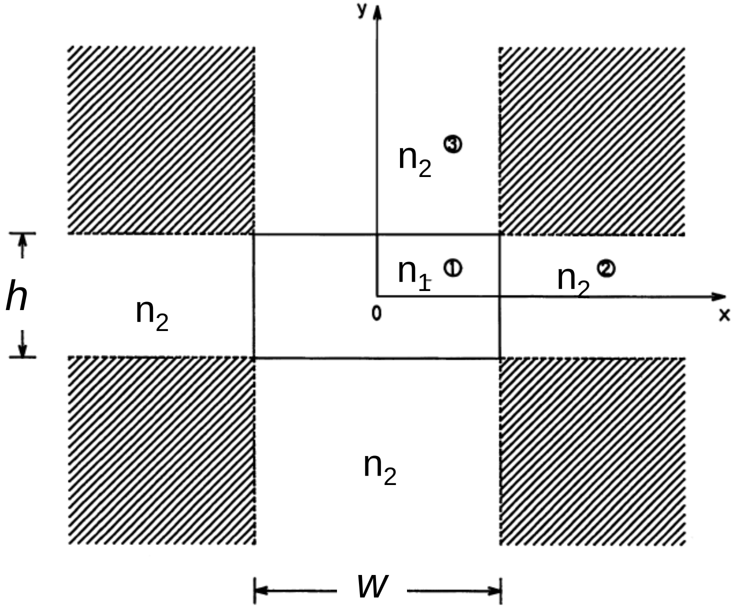


Slab waveguide solver: http://www.computational-photonics.eu/oms.html

Accuracy: Comparison of different methods



Chen, Foundations for Guided Wave Photonics



- Semi-analytical methods are accurate far above cutoff.
- Near cutoff, the fields are not well confined to the core; the basic assumptions for the analytical methods are hence not any more fulfilled.
- Marcatili's method tends to underestimate the propagation constant, whereas the effective-index method tends to overestimate it.

Numerical Mode Solvers

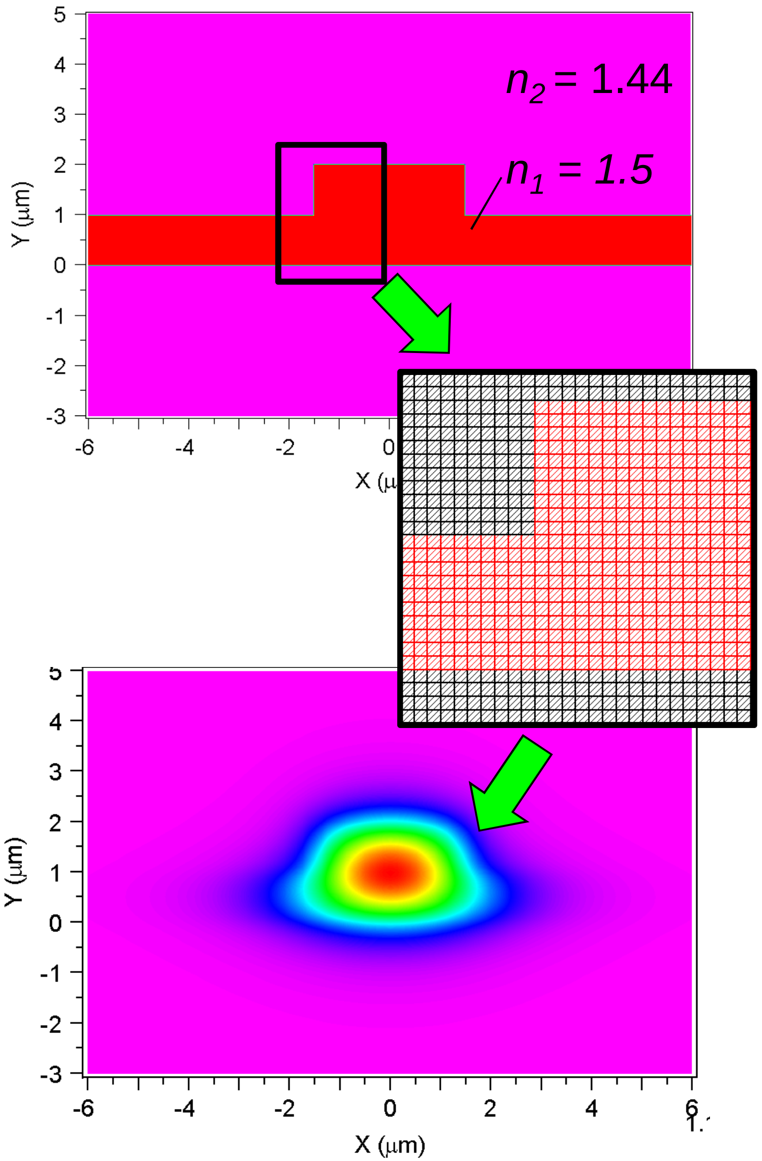
General procedure:

- Choice of a finite computational domain
- Discretization of refractive index profile
- Discretization of field equations
- Numerical solution of resulting linear system of equations / eigenvalue problem

Numerical mode solvers come with most commercial software packages for electromagnetic field simulations:

- Rsoft, Ossining, NY, *BEAMProp,* *http://www.rsoftdesign.com*
- CST, Darmstadt, *Microwave Studio*, http://www.cst.de
- Photon Design, Oxford, UK, *FIMMwave* *http://www.photond.com/*
- Ansoft, Pittsburgh, *HFSS* *http://www.ansoft.com*

… and many more …



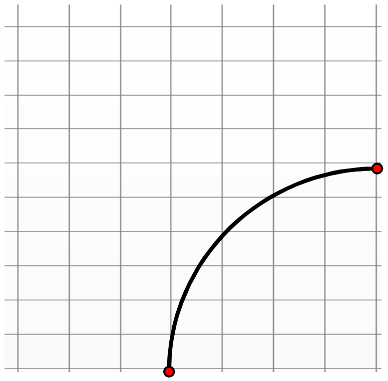
Finite-Difference Method

- Start from vectorial or scalar mode field equation (for weakly inhomogeneous media)

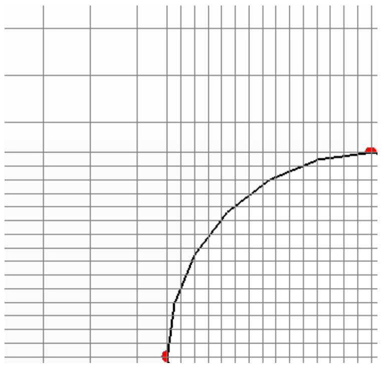
$$\frac{\partial^2 \underline{\Psi}(x,y)}{\partial x^2} + \frac{\partial^2 \underline{\Psi}(x,y)}{\partial y^2} + \left(k_0^2 n^2(x,y) - \beta^2\right) \underline{\Psi}(x,y) = 0$$

$\underline{\Psi}(x,y)$ can be any component of the mode fields $\underline{\mathcal{E}}(x,y)$ and $\underline{H}(x,y)$.

- Refractive index profile is sampled at discrete grid points that may or may not be equidistant



Uniform grid:
Δ x, Δ y do not vary throughout the computational domain



Nonuniform grid:
Δ x, Δ y are locally adapted to the structure

- Derivatives are replaced by finite differences

$$f'(x_p) \approx \frac{f(x_{p+1}) - f(x_{p-1})}{2\Delta x}, \qquad f''(x_p) \approx \frac{f(x_{p+1}) + f(x_{p-1}) - 2f(x_p)}{\Delta x^2}$$

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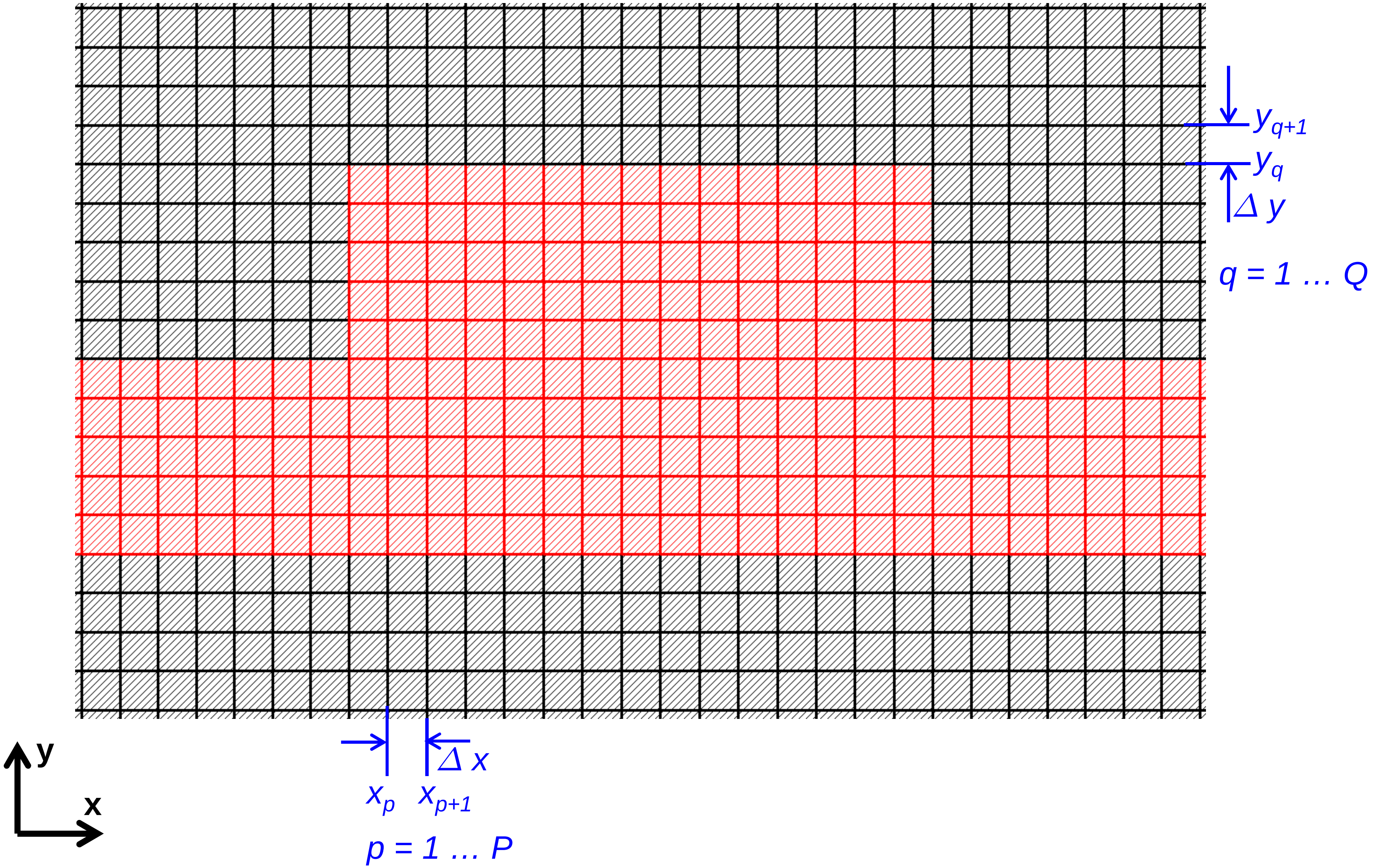
Finite-Difference Method



Finite-Difference Method

Discrete approximation of second-order partial derivatives:

$$\left.\frac{\partial^2 \underline{\Psi}(x,y)}{\partial x^2}\right|_{x=x_p,\, y=y_q} = \frac{\underline{\Psi}_{p+1,q} + \underline{\Psi}_{p-1,q} - 2\underline{\Psi}_{p,q}}{\Delta x^2},$$

$$\left.\frac{\partial^2 \underline{\Psi}(x,y)}{\partial y^2}\right|_{x=x_p,\, y=y_q} = \frac{\underline{\Psi}_{p,q+1} + \underline{\Psi}_{p,q-1} - 2\underline{\Psi}_{p,q}}{\Delta y^2}$$

Discretized mode field equation :

$$\frac{\underline{\Psi}_{p+1,q} + \underline{\Psi}_{p-1,q} - 2\underline{\Psi}_{p,q}}{\Delta x^2} + \frac{\underline{\Psi}_{p,q+1} + \underline{\Psi}_{p,q-1} - 2\underline{\Psi}_{p,q}}{\Delta y^2} + k_0^2 n_{p,q}^2 \underline{\Psi}_{p,q} = \beta^2 \underline{\Psi}_{p,q}$$

Formulation as a linear eigenvalue problem:

$$\mathbf{A}\underline{\widehat{\Psi}} = \beta^2 \underline{\widehat{\Psi}}$$

$\Rightarrow$ **To be solved numerically**

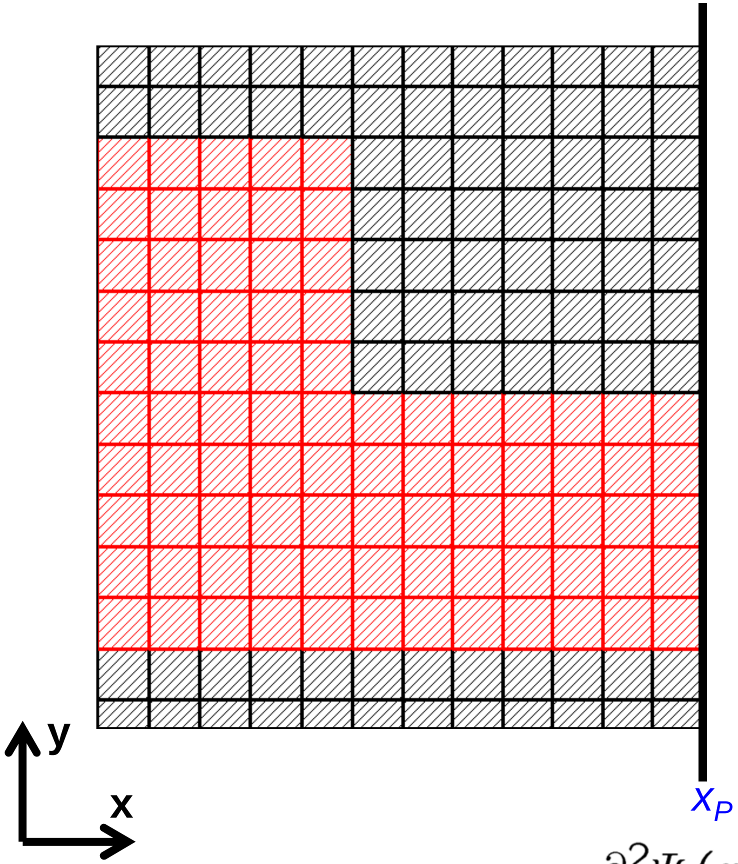
[PQ,PQ]-matrix

[PQ,1]-vector
(eigenvector)

Propagation constant
(eigenvalue)

Termination of computational domain

Problem: Field quantities at the edges of the computational domain are related to unknown field quantities outside this area.



Note: Simply setting $\Psi_{P+1,q} = 0$ is not a good solution! This would be equivalent to terminating the computational domain with a perfect (metallic) reflector!

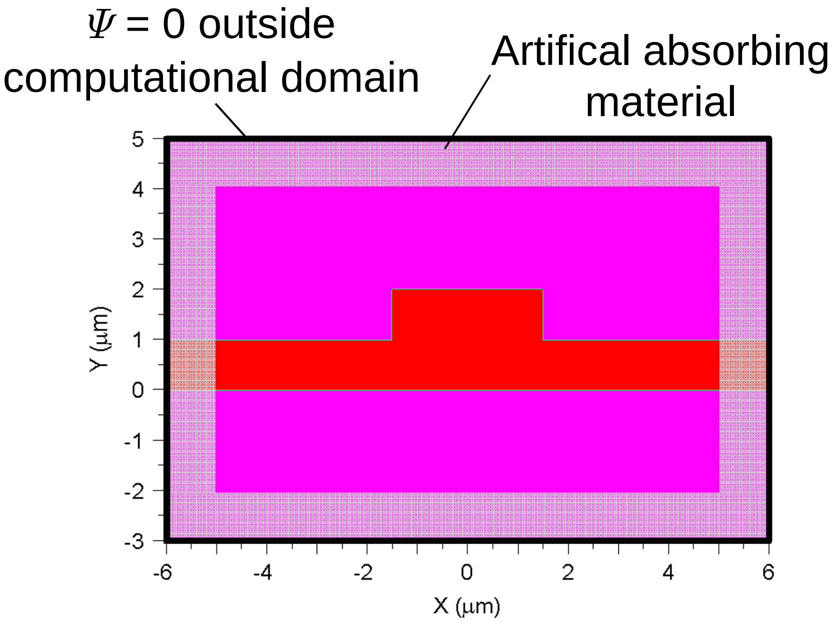
Goal: Domain boundary should be transparent for outgoing waves!

Outside the computational domain!

$$\left.\frac{\partial^2 \underline{\Psi}(x,y)}{\partial x^2}\right|_{x=x_P,\, y=y_q} = \frac{\underline{\Psi}_{P+1,q} + \underline{\Psi}_{P-1,q} - 2\underline{\Psi}_{P,q}}{\Delta x^2}$$

Termination of computational domain

Perfectly matched layers (PML)

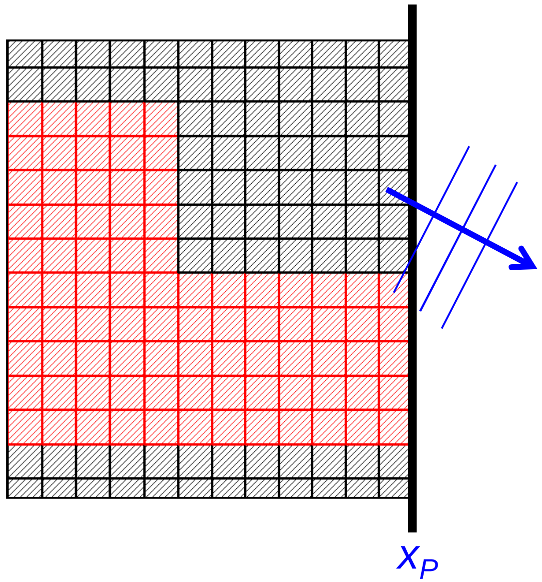


Note: Wave impedance of absorber must be matched to impedance of the computational domain to prevent back-reflection from the surface

Drawbacks:

- Adjustment of PML parameters crucial
- Requires extension of computational domain to incorporate PML

Transparent boundary conditions (TBC)



- **Assumption:** Field near the boundary behaves like an outgoing plane wave
- Parameters (amplitude, direction) are determined via some heuristic algorithm
- Plane-wave assumption allows to estimate the field values outside the computational domain.

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Finite-Element Method

The finite element method does not start from the wave equations directly, but from a related problem that is based on the minimization of an integral expression over the computational domain.

Example: It can be shown* that solving the scalar wave equation,

$$\frac{\partial^2 \underline{\Psi}(x,y)}{\partial x^2} + \frac{\partial^2 \underline{\Psi}(x,y)}{\partial y^2} + \left(k_0^2 n^2(x,y) - \beta^2\right) \underline{\Psi}(x,y) = 0$$

is mathematically equivalent to minimizing a functional of the form

$$I = \iint_\Omega \left[\left(\frac{\partial \underline{\Psi}(x,y)}{\partial x} \right)^2 + \left(\frac{\partial \underline{\Psi}(x,y)}{\partial y} \right)^2 + \left(k_0^2 n^2(x,y) - \beta^2\right) \underline{\Psi}^2(x,y) \right] \mathrm{d}x \, \mathrm{d}y$$
$$\underbrace{- \int_{\partial\Omega} \left[\underline{\Psi}(x,y) \frac{\partial \underline{\Psi}(x,y)}{\partial n} \right] \mathrm{d}s}_{=0} \rightarrow \text{Min!}$$

= 0
for Dirichlet or Neumann boundary conditions

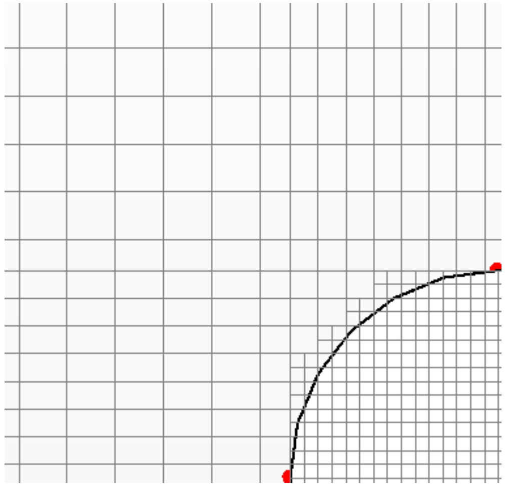
*Kawano, Kitoh et al., Optical Waveguide Analysis, Wiley, 2001

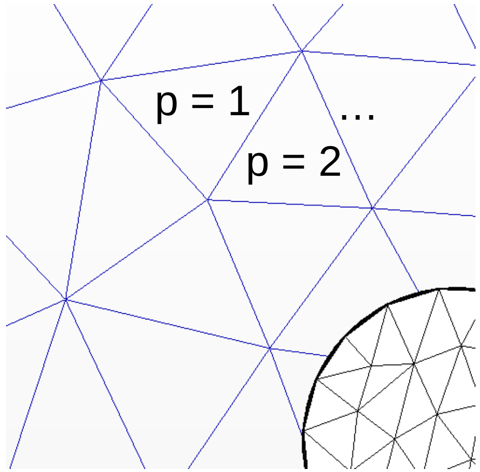
Finite-Element Method

Further steps:

- Discretization of computational domain in P elements (triangles, rectangles …)
- Field expansion in each of the elements into q = 1 … Q basis functions $\Psi_{pq}(x,y)$

$$\Psi_p(x,y) = \sum_q c_{pq} \Psi_{pq}(x,y)$$





- The functional can then be written as quadratic matrix equation

  c: (PQ,1) - vector of coefficients c_{pq}
  A, **B**: (PQ,PQ) – matrices (Contains integrals of basis functions over elements)

$$I = \frac{1}{2}\mathbf{c}^{\mathrm{T}}\left(\mathbf{A} - \left(k_0^2 n^2 - \beta^2\right)\mathbf{B}\right)\mathbf{c} \rightarrow \text{Min!}$$

- The stationarity condition for the functional yields an eigenvalue matrix equation that can be solved numerically to obtain the expansion coefficients c_{pq} and the propagation constant β

$$\nabla_{\mathbf{c}}^T I = \left(\mathbf{A} - \left(k_0^2 n^2 - \beta^2\right)\mathbf{B}\right)\mathbf{c} = 0 \qquad \Rightarrow \mathbf{B}^{-1}\mathbf{A}\mathbf{c} = \left(k_0^2 n^2 - \beta^2\right)\mathbf{c}$$

Eigenvector (expansion coefficients)

Eigenvalue (leads to propagation constant)

Numerical Mode Solvers

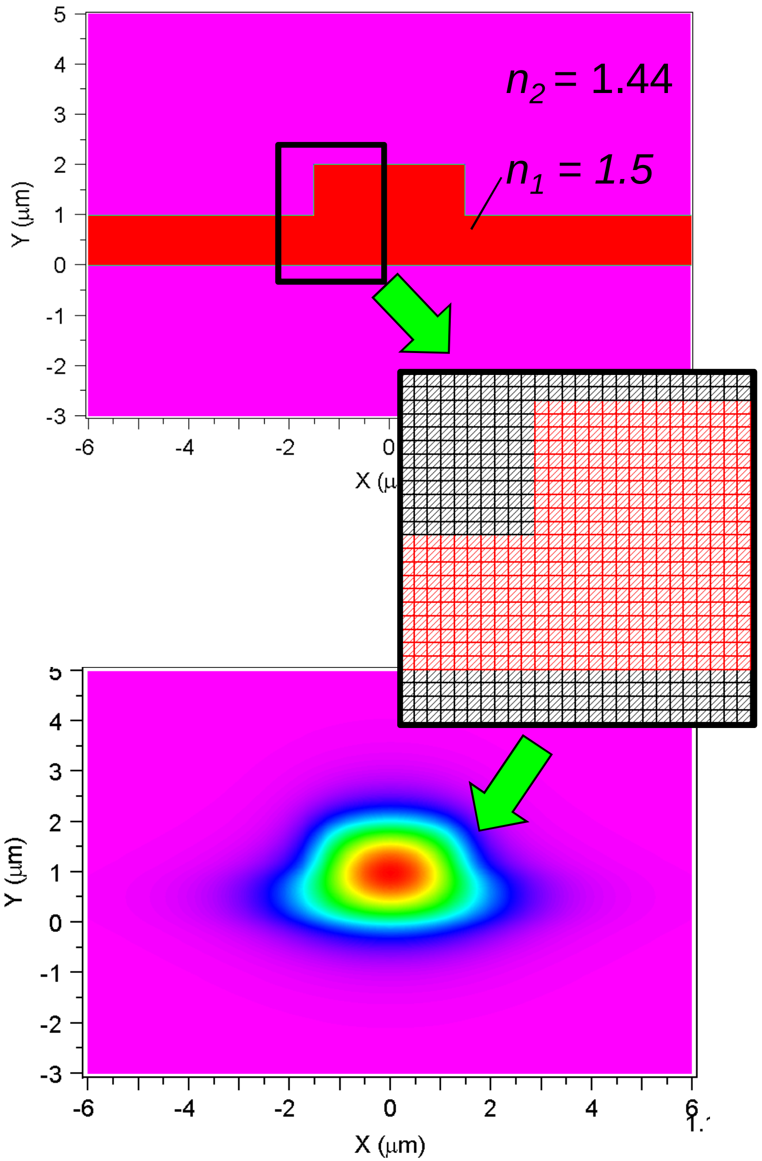
General procedure:

- Choice of a finite computational domain
- Discretization of refractive index profile
- Discretization of field equations
- Numerical solution of resulting linear system of equations / eigenvalue problem

Numerical mode solvers come with most commercial software packages for electromagnetic field simulations:

- Rsoft, Ossining, NY, *BEAMProp,* *http://www.rsoftdesign.com*
- CST, Darmstadt, *Microwave Studio*, http://www.cst.de
- Photon Design, Oxford, UK, *FIMMwave* *http://www.photond.com/*
- Ansoft, Pittsburgh, *HFSS* *http://www.ansoft.com*

… and many more …



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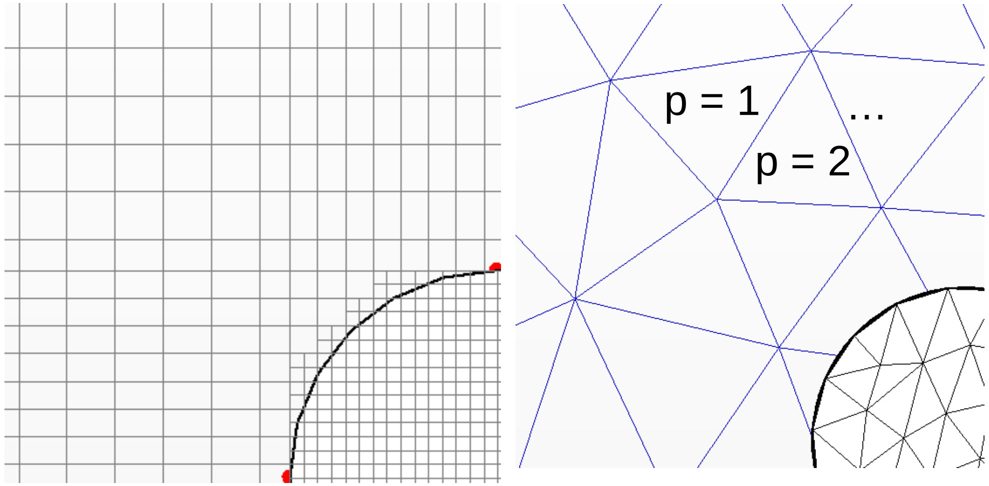


- The functional can then be written as quadratic matrix equation

  c: (PQ,1) - vector of coefficients c_{pq}

  A, **B**: (PQ,PQ) – matrices (Contains integrals of basis functions over elements)

$$I = \frac{1}{2}\mathbf{c}^{\mathrm{T}}\left(\mathbf{A} - \left(k_0^2 n^2 - \beta^2\right)\mathbf{B}\right)\mathbf{c} \rightarrow \text{Min!}$$

- The stationarity condition for the functional yields an eigenvalue matrix equation that can be solved numerically to obtain the expansion coefficients c_{pq} and the propagation constant β

$$\nabla_{\mathbf{c}}^T I = \left(\mathbf{A} - \left(k_0^2 n^2 - \beta^2\right)\mathbf{B}\right)\mathbf{c} = 0 \quad \Rightarrow \mathbf{B}^{-1}\mathbf{A}\mathbf{c} = \left(k_0^2 n^2 - \beta^2\right)\mathbf{c}$$

Eigenvector (expansion coefficients) → $\mathbf{c}$

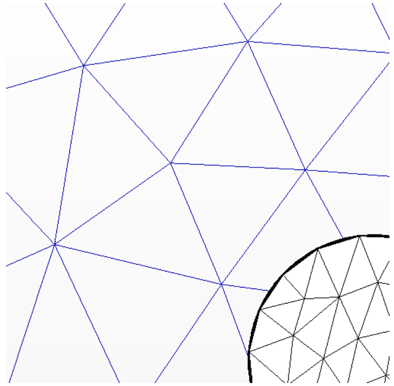
Eigenvalue (leads to propagation constant) → $\left(k_0^2 n^2 - \beta^2\right)$

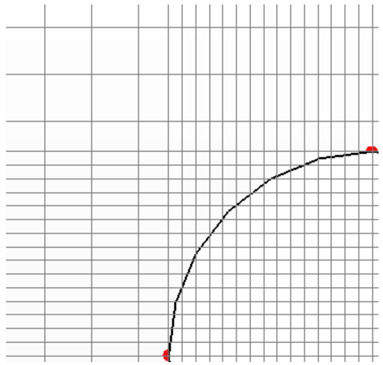
Sources of errors in numerical methods

- **Model errors** due to basic assumptions in the underlying algorithm, e.g.,
  - **Index contrast:** Methods for low-index contrast fibers cannot necessarily be applied to high-index contrast integrated waveguides!
  - **Vectorial vs. scalar approaches:** Scalar methods should only be used in weakly guiding waveguides

- **Discretization errors**
  - Representation of refractive index profile by discrete grid points
  - Finite difference approximation of the derivatives / finite element approximation of an integral expression
  - Note: Given a certain number of grid points / finite elements, FEM methods usually allow for better representation, since elements adapt to structure shape

  $\Rightarrow$ **Refine mesh and check convergence**



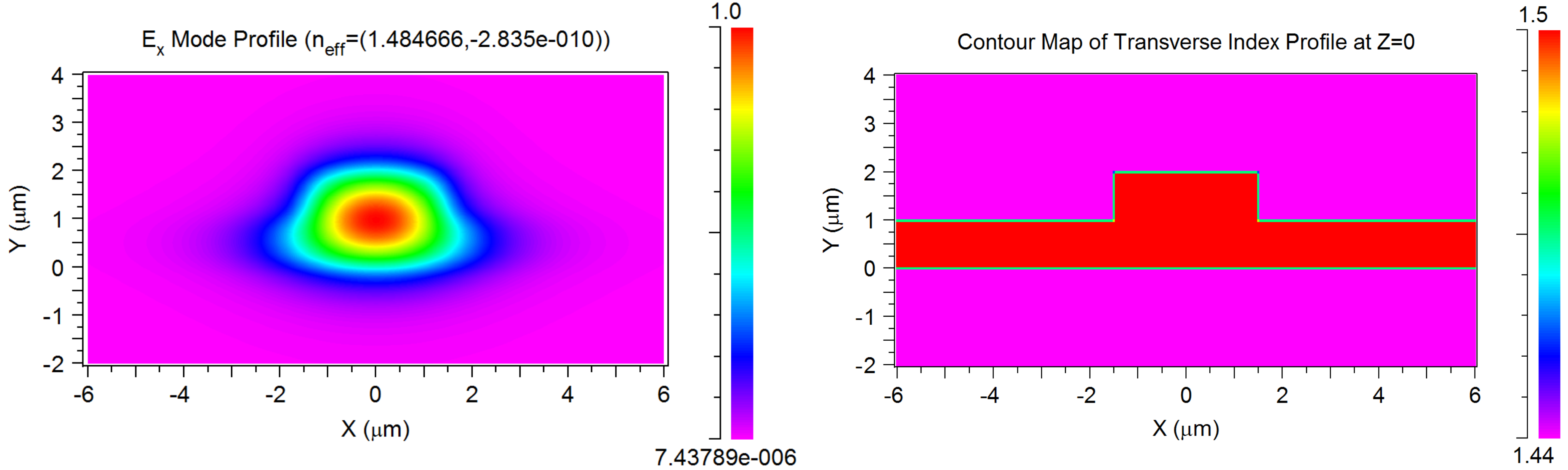


- **Finite computational domain / boundary conditions at domain edges**

  $\Rightarrow$ **Extend computational domain / PML width and check convergence**

Demonstration: Numerical calculation of rib wavguide mode

Program: Rsoft FemSIM

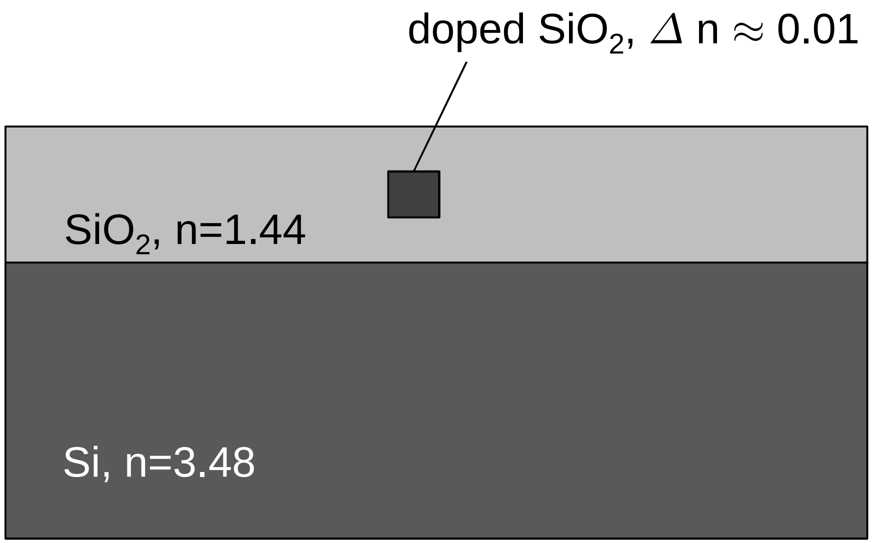


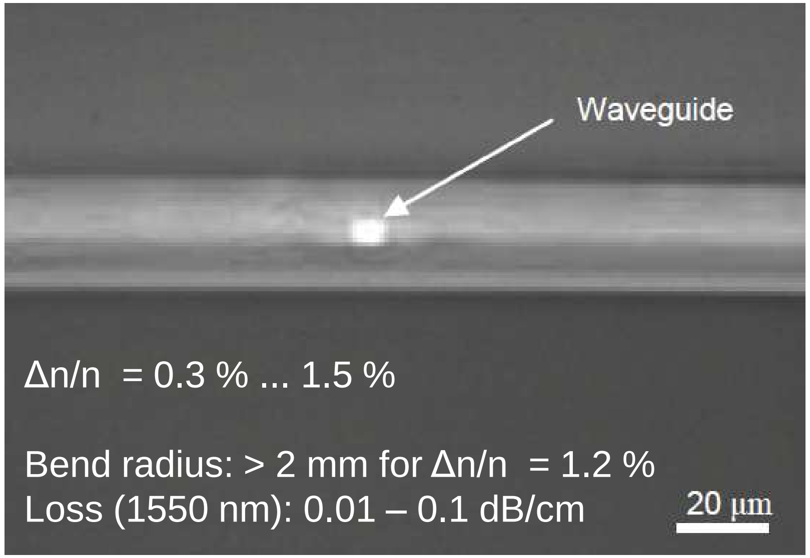
Planar waveguide technologies – an overview

Mainstream technologies for planar lightwave circuits (PLC):

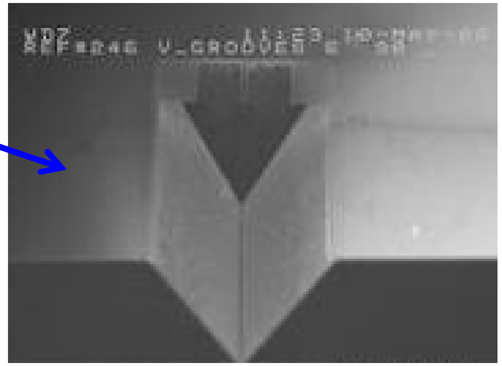
- Glass waveguides
  Low index-contrast, fabricated by ion exchange or deposition and etching
- Lithium niobate waveguides
  Used mainly for electro-optic modulators
- Polymer waveguides
  Easy fabrication, but absorption losses in infrared
- Silicon nitride / Triplex waveguides
  Variable index contrast, low loss
- Waveguides based on III-V compound semiconductors
  Used for active devices (lasers, semiconductor optical amplifiers, photodetectors)
- Silicon-on-insulator (SOI) waveguides
  Very compact, fabrication in CMOS fabs, current area of research

Glass waveguides fabricated by deposition / etching; Silicon optical bench (SOB)





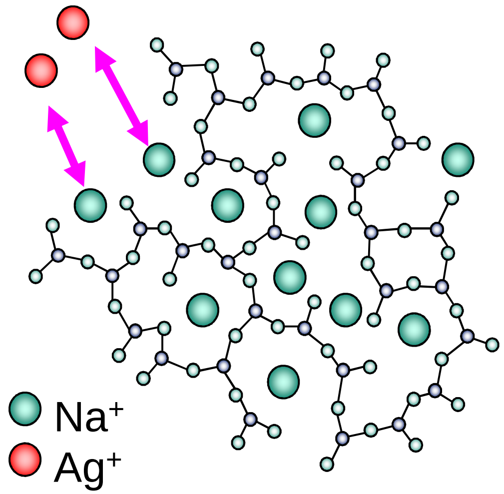
- **Silicon substrate:** Widely available, good thermal sink, good mechanical alignment (V-grooves for fibers)
- **Geometry:** Typical core size is 5 x 5 µm, the core is surrounded by ~ 20 µm base and cladding layers; relative index difference Δ between 0.3 % and 1.5 %
- **Fabrication:**
  - The thick silica film is formed by chemical vapor deposition (CVD) or flame hydrolysis deposition (FHD) of SiO_2
  - Core layer is deposited by CVD and structured by photolithography and reactive ion etching (RIE).
  - Cladding is deposited by CVD
- **Applications:** Optical communications, chemical sensing

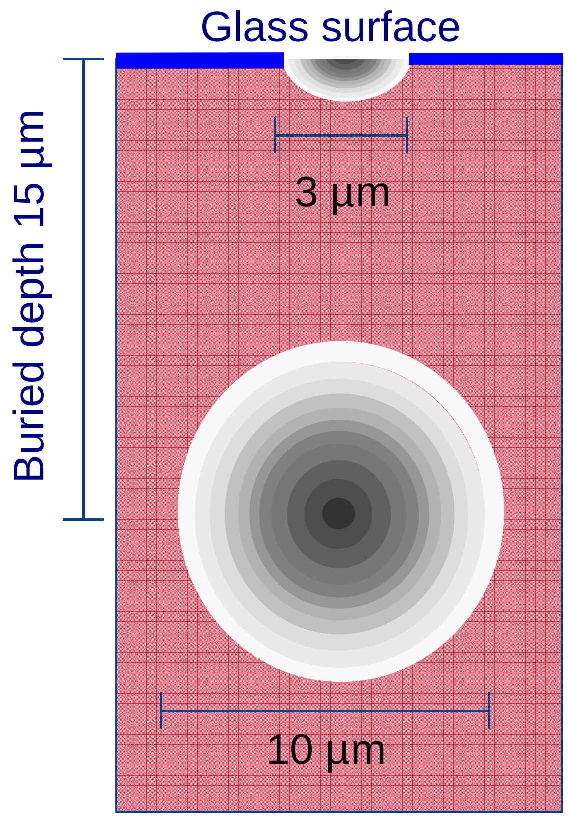


Glass waveguides based on ion exchange

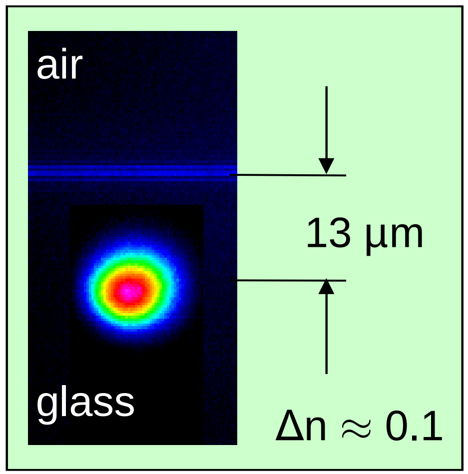
Principle:

- Refractive index of speciality glasses (e.g., IAG4) can be increased by ion exchange, i.e., by substituting Na with Ag
- Two-step process to fabricate embedded waveguides:





1. After thermal diffusion (first step)
2. After E-field diffusion (second step)



Source: Leoni

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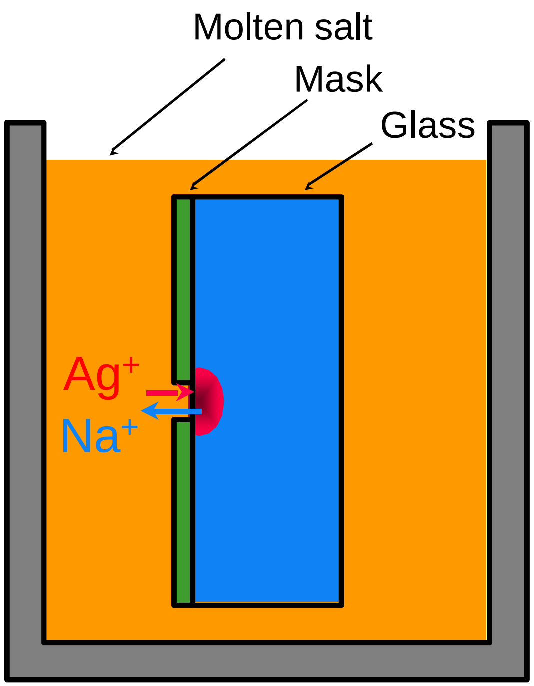
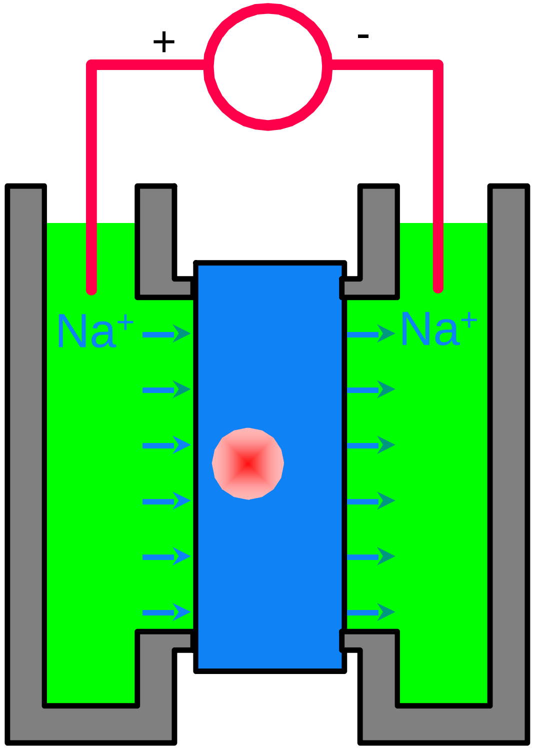
Glass waveguides based on ion exchange



Step 1:
Waveguide near surface,
fabricated by thermal diffusion



Step 2:

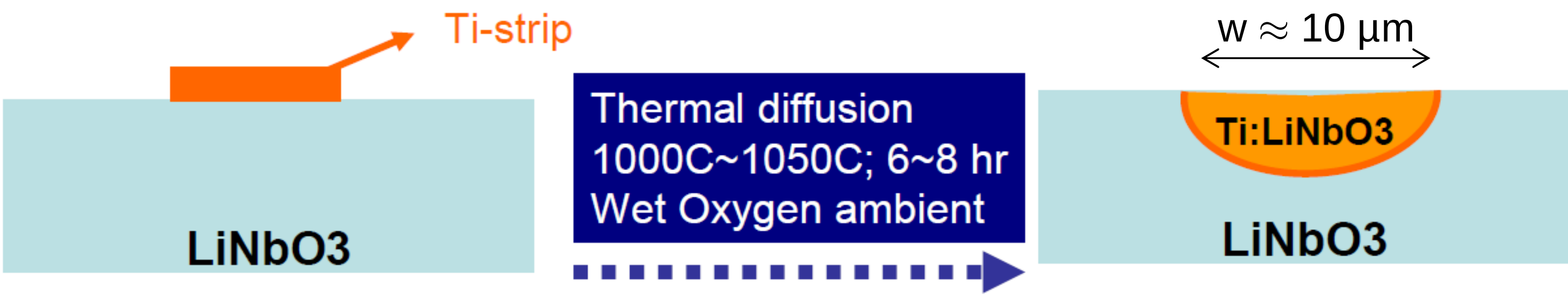
Buried waveguide,
fabricated by field-assisted ion exchange

Source: Leoni

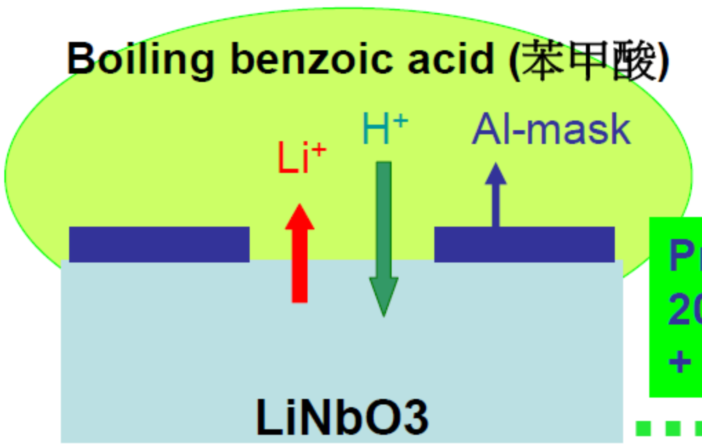
$LiNbO_3$ waveguides: Ti-indiffusion or proton exchange

- **Ti-indiffused (TI) waveguides**

$\Delta n \approx 0.002 \ldots 0.01$



- **Proton-exchanged (PE) waveguides**



$\Delta n \approx 0.02 \ldots 0.05$ (extraordinary pol)
$\Delta n \approx -0.04$ (ordinary pol)

Polymers: Absorption Properties

- Relatively low loss at ~ 800 nm, but very high losses at 1300 nm and 1500 nm
- **Reason:** Overtones of C-H-bond oscillations: Fundamental oscillation at 3390 nm; overtone at ~1700 nm, 850 nm …; absorption decreases exponentially with decreasing wavelength!
- Reduction of losses if H is substituted by Deuterium (D), Cl or F (larger atom mass leads to lower oscillation frequency, i.e. absorption is shifted further into IR)

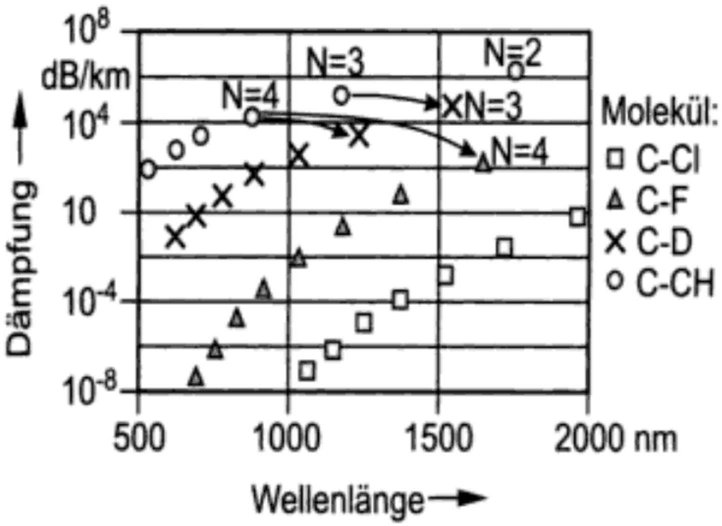
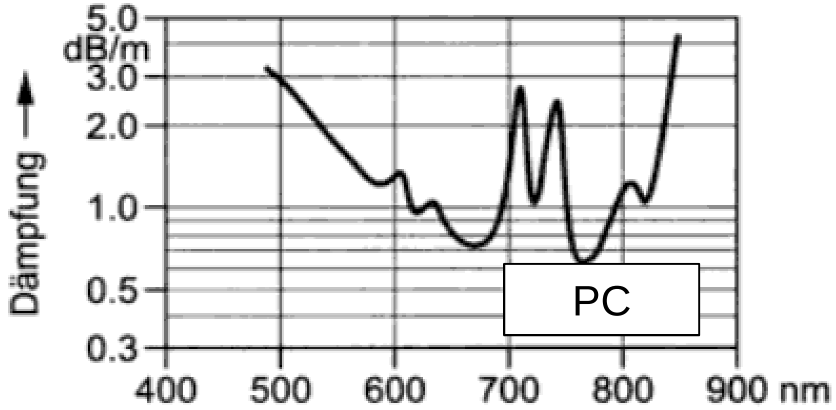
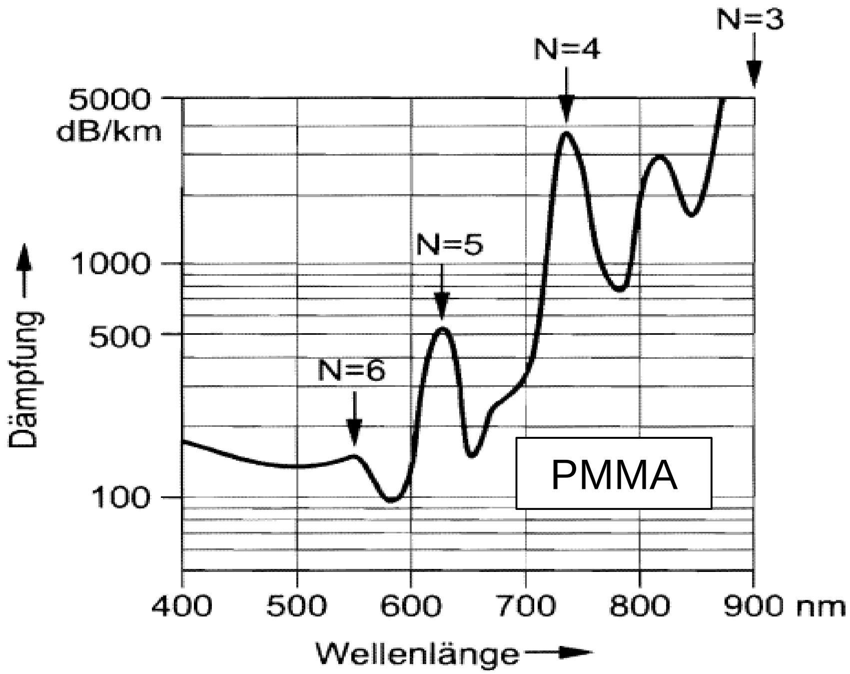


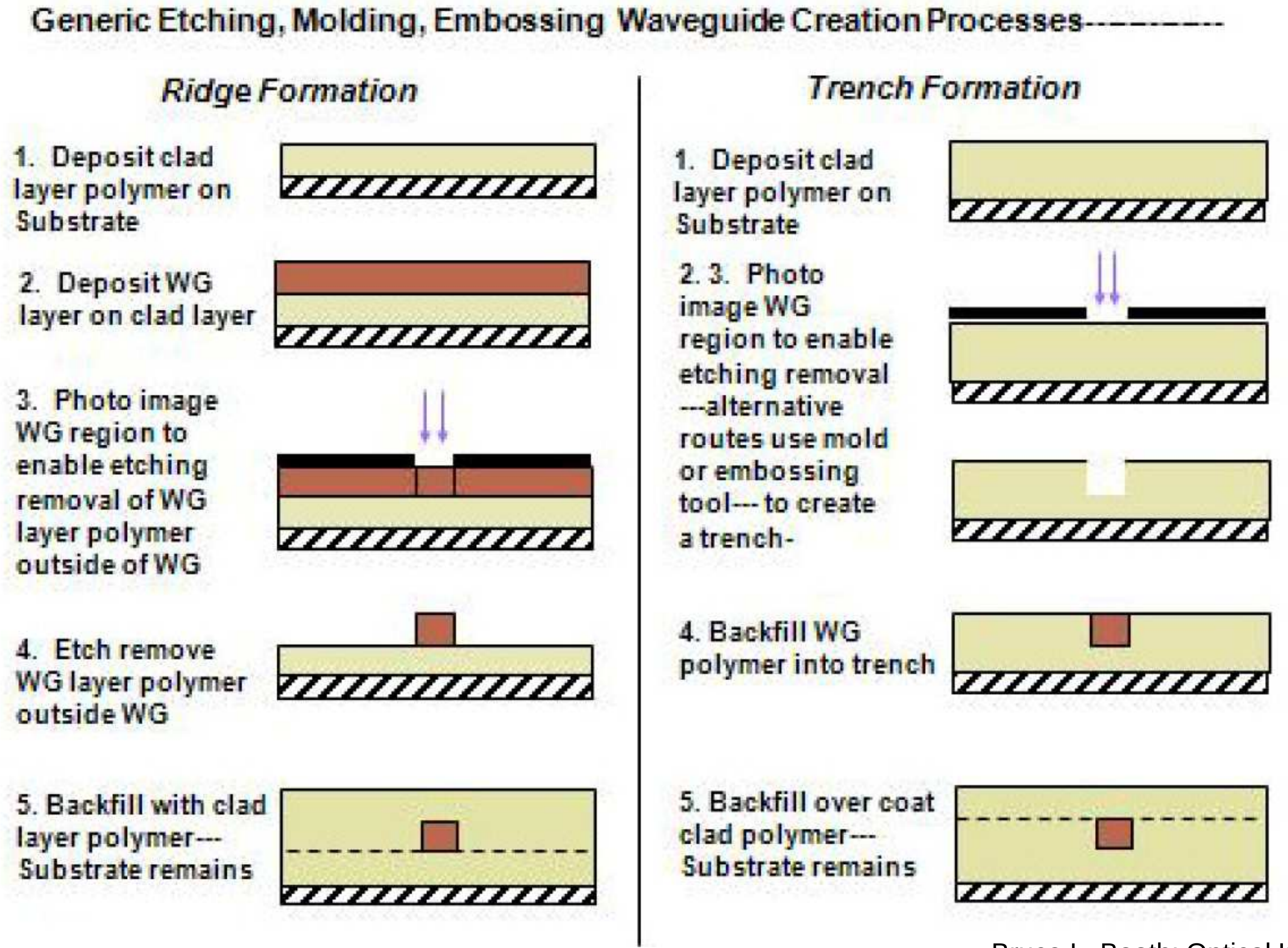
Abb. 8.7. Dämpfungsbeiträge der Obertöne von CH-, CD-, CF- und Cl-Bindungen in Polymeren





Voges/ Petermann: Optische Kommunikationstechnik

Polymer waveguides

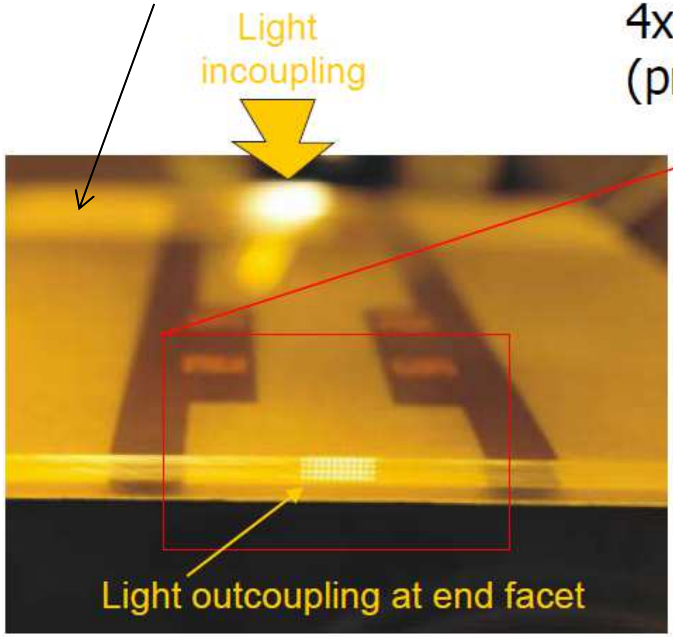


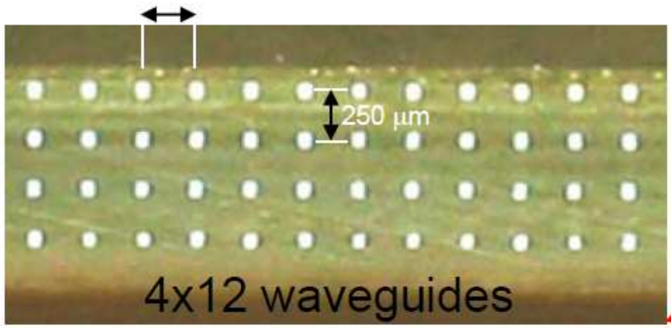
Bruce L. Booth; Optical InterLinks, LLC

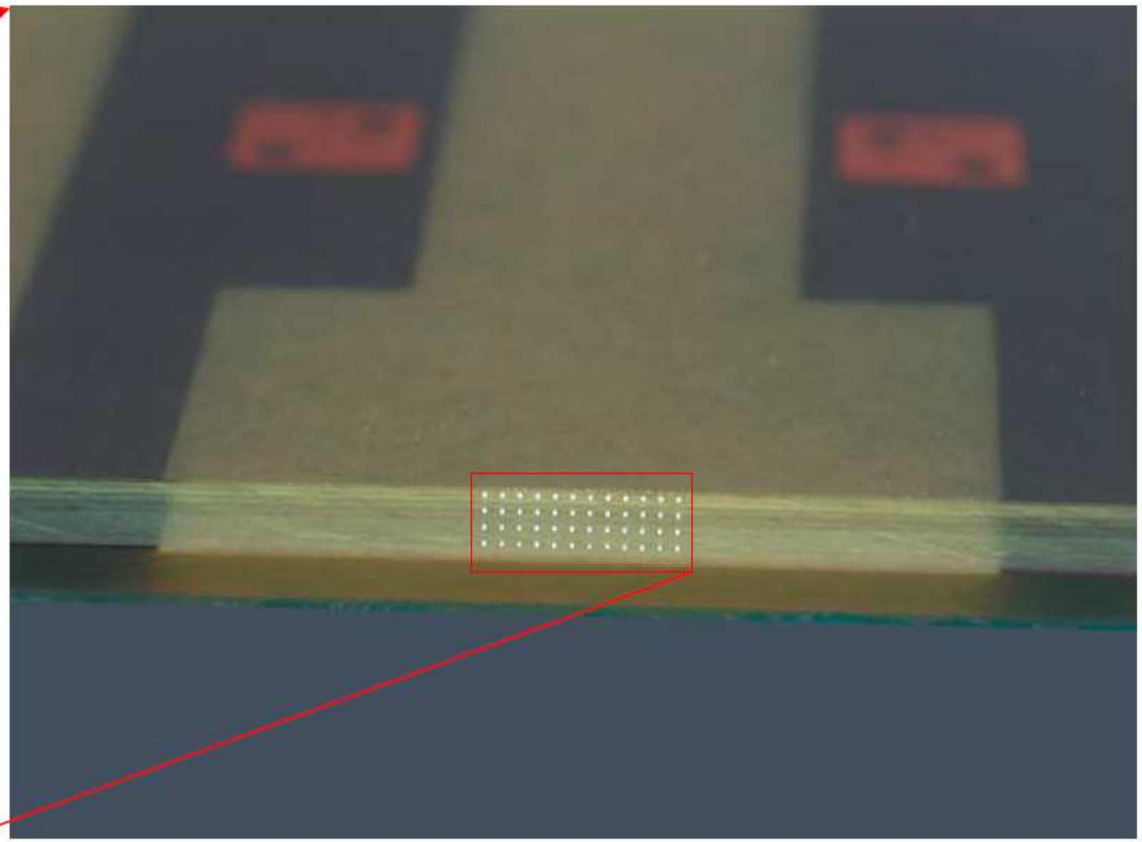
Polymer waveguides

Printed circuit board (PCB)

4x12 waveguides on 250μmx250μm grid
(proof of feasibility)

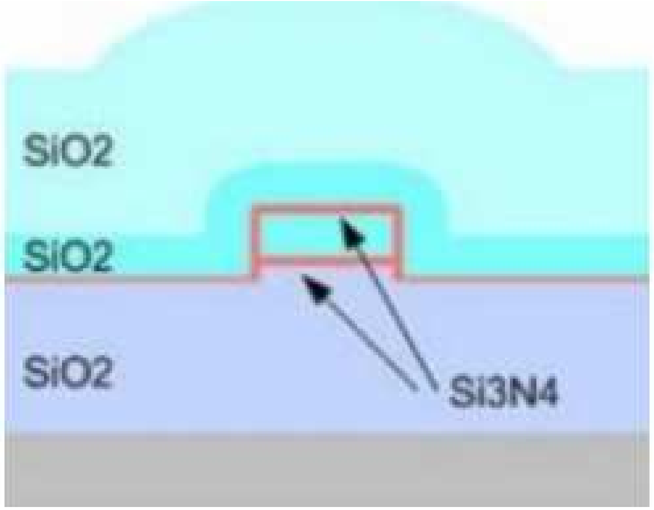


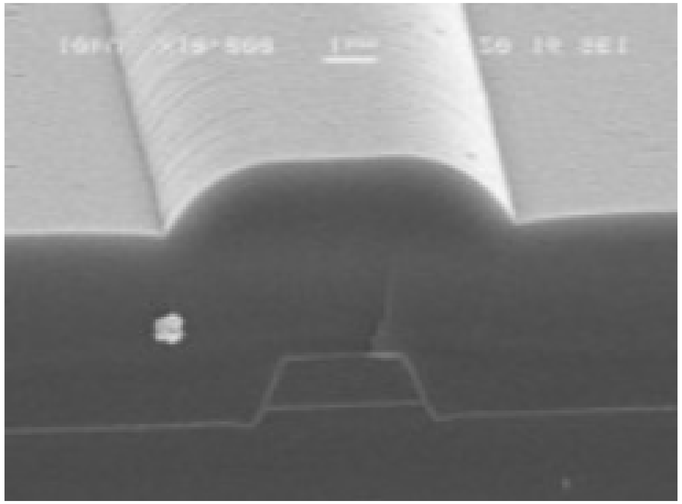




Yurii Vlasov, 'Silicon photonics for next generation computing systems', ECOC 2008

Si_3N_4 / SiO_2 waveguides





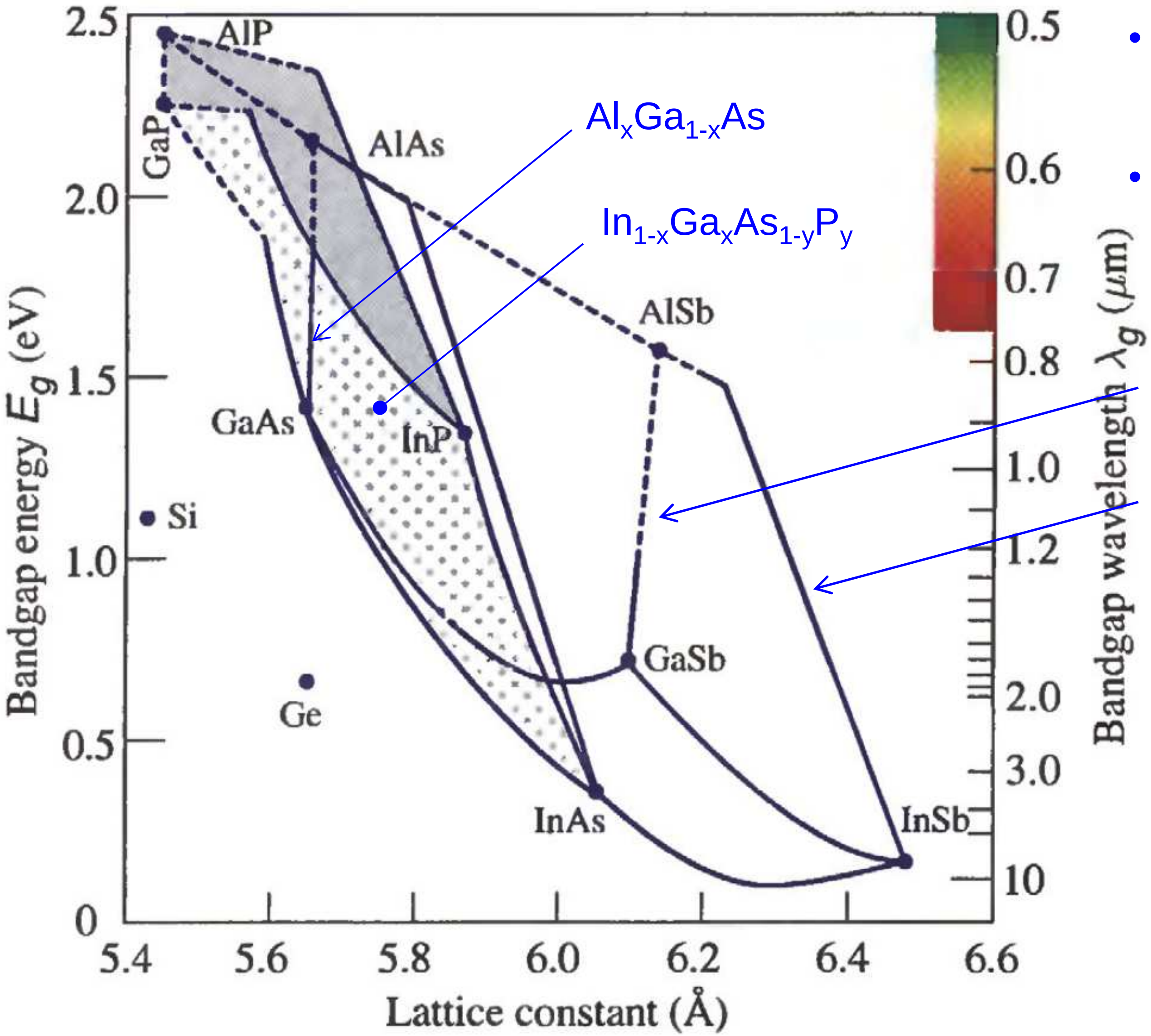
- **Refractive indices:**
  SiO_2: n = 1.44 at 1.55 µm
  Si_3N_4: n ≈ 2.1 at 1.55 µm
  SiO_xN_y: n = 1.44 … 2.1 at 1.55 µm

- **Waveguide core**:
  - Low-index silicon dioxide (SiO_2) surrounded by a thin film of a high-index silicon nitride (Si_3N_4); index contrast can be adjusted by thickness of Si_3N_4 layer
  - Low losses: <0.1dB/cm
  - Typical core size: 1um x 1um

- **Applications:** Datacom, chemical sensing.

LioniX trademark: "Triplex"
http://www.lionixbv.nl/download/pdf/flyertriplex.pdf

Active waveguides: III-V compound semiconductors



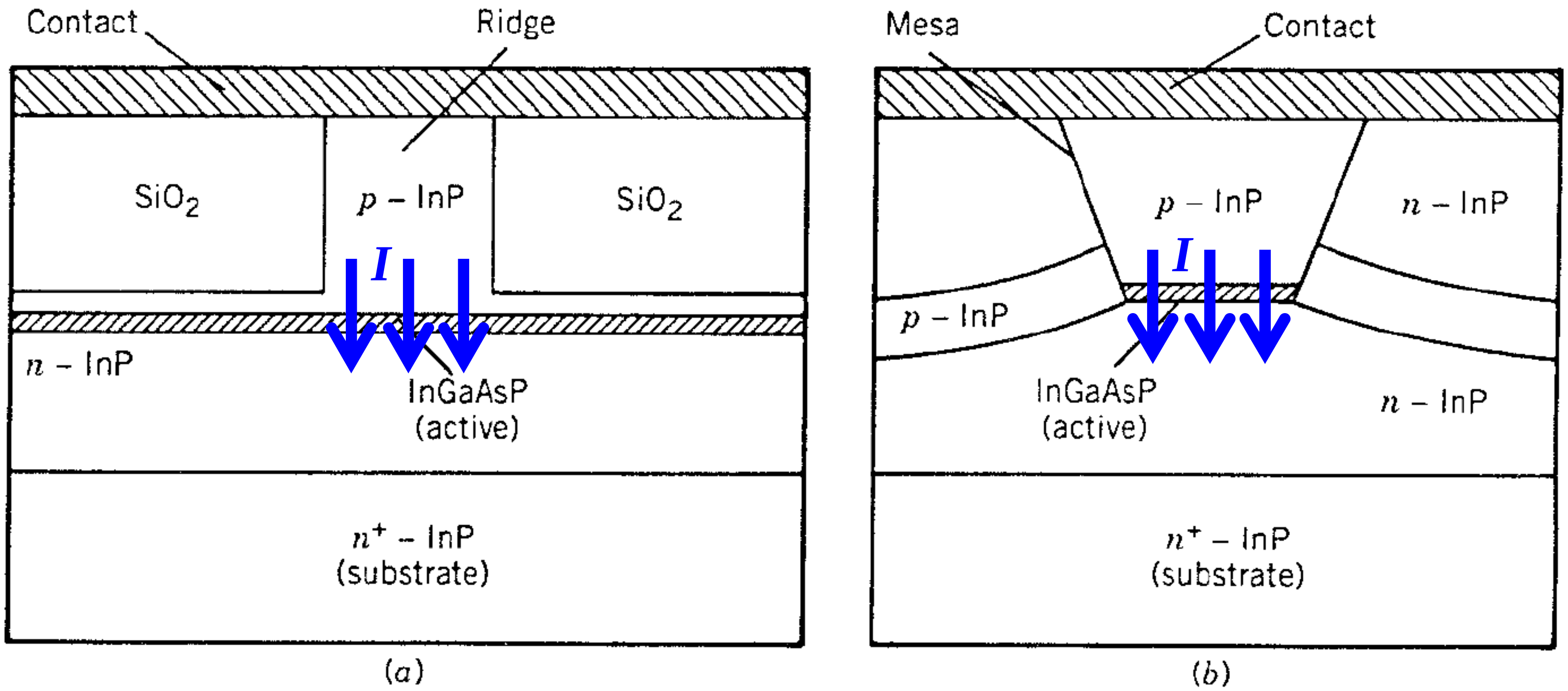
- Ternary compounds represented by the line that joins two binary compounds
- Quaternary compounds represented by the area defined by binary compound corners

Dashed lines: Indirect bandgap

Solid lines: Direct bandgap

Source: Saleh/Teich, Fundamentals of Photonics

Active waveguides: III-V compound semiconductors



Cross sections of semiconductor lasers: Ridge and channel waveguides

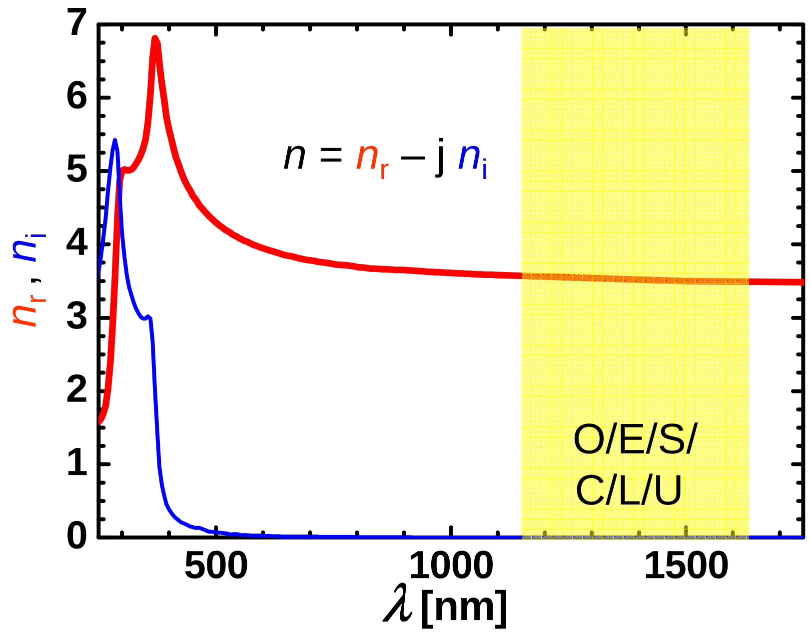
- Refractive indices:

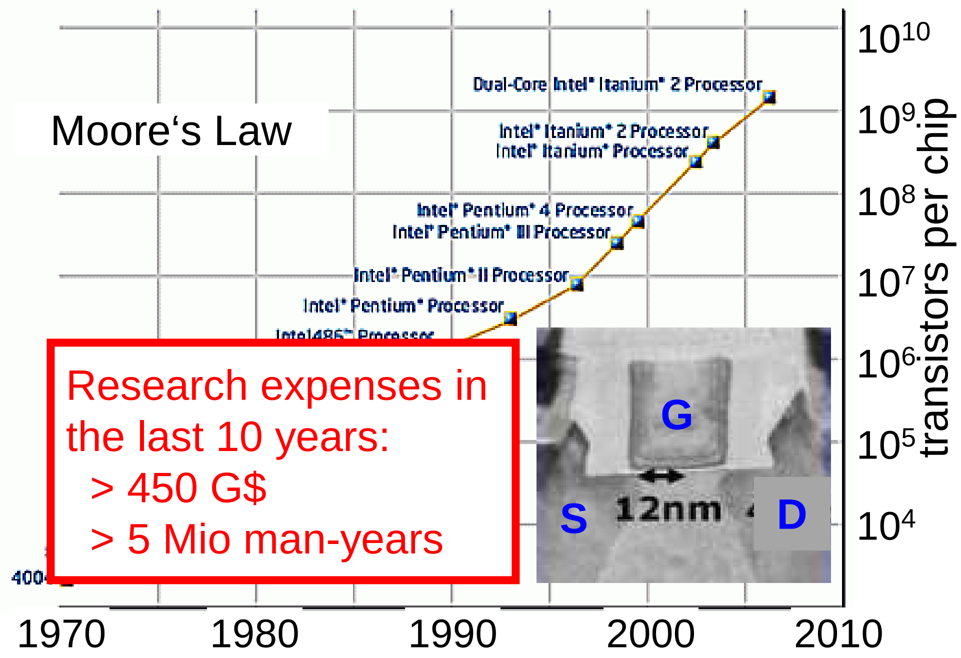
InP:	n = 3.17 at 1550 nm
InGaAsP:	n $\approx$ 3.4 … 3.6 at 1550 nm
SiO_2:	n = 1.44 at 1550 nm

Silicon photonics

Silicon:

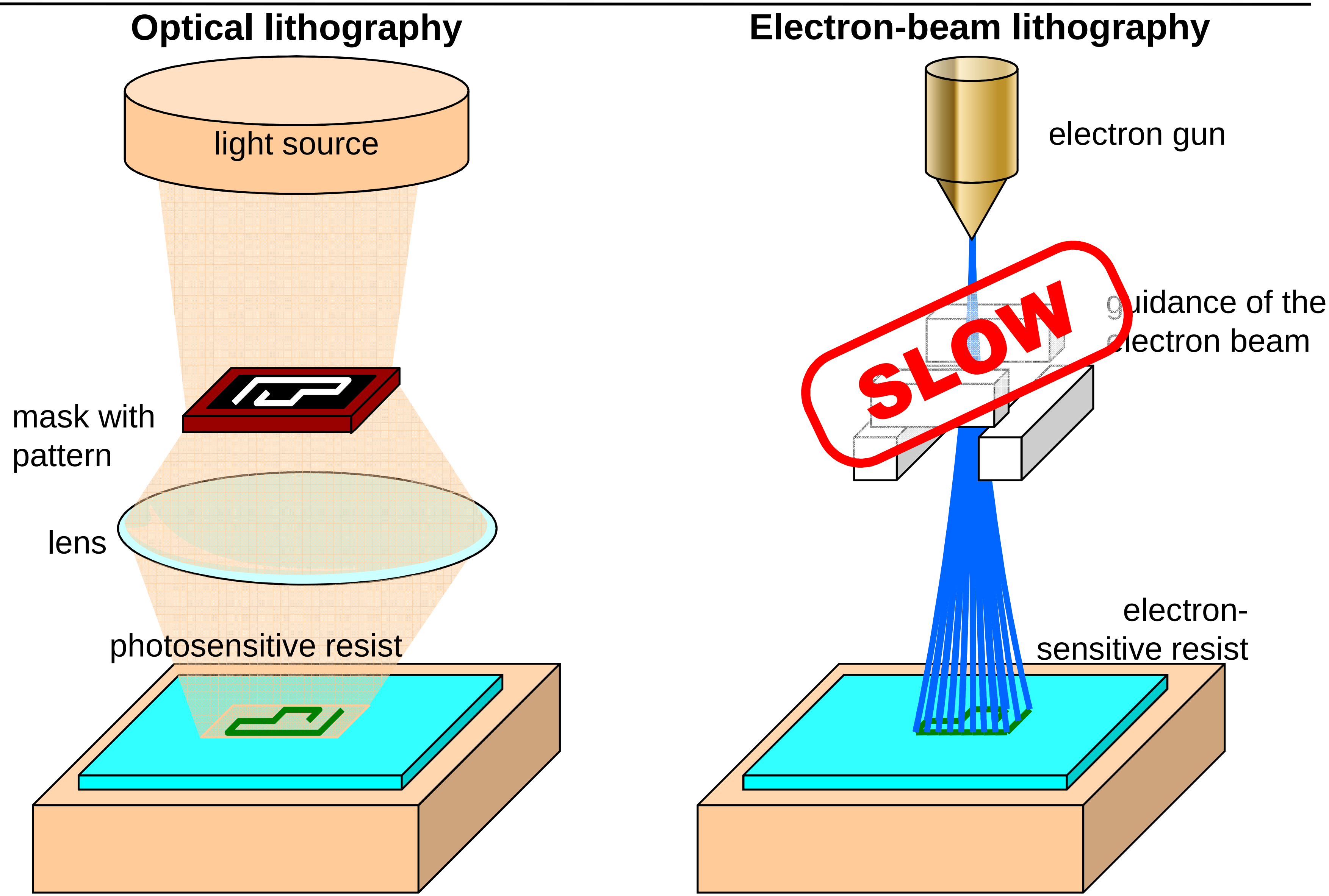
- Transparent at IR telecom wavelengths
- High refractive index
  - ⇒ Nanophotonic devices
    Dense integration
    Small active volumes
    Ultra-fast low-power switching





- Mature CMOS fabrication processes
  - ⇒ Cheap mass-production of nanophotonic devices
  - ⇒ Full integration of photonics and electronics

Crucial: Fast high-resolution lithography



Animation courtesy of Prof. Roel Baets, UGENT/IMEC

State-of-the art CMOS tool: 193nm Immersion Lithography

$$\text{res} = k_1 \frac{\lambda}{\text{NA}}$$

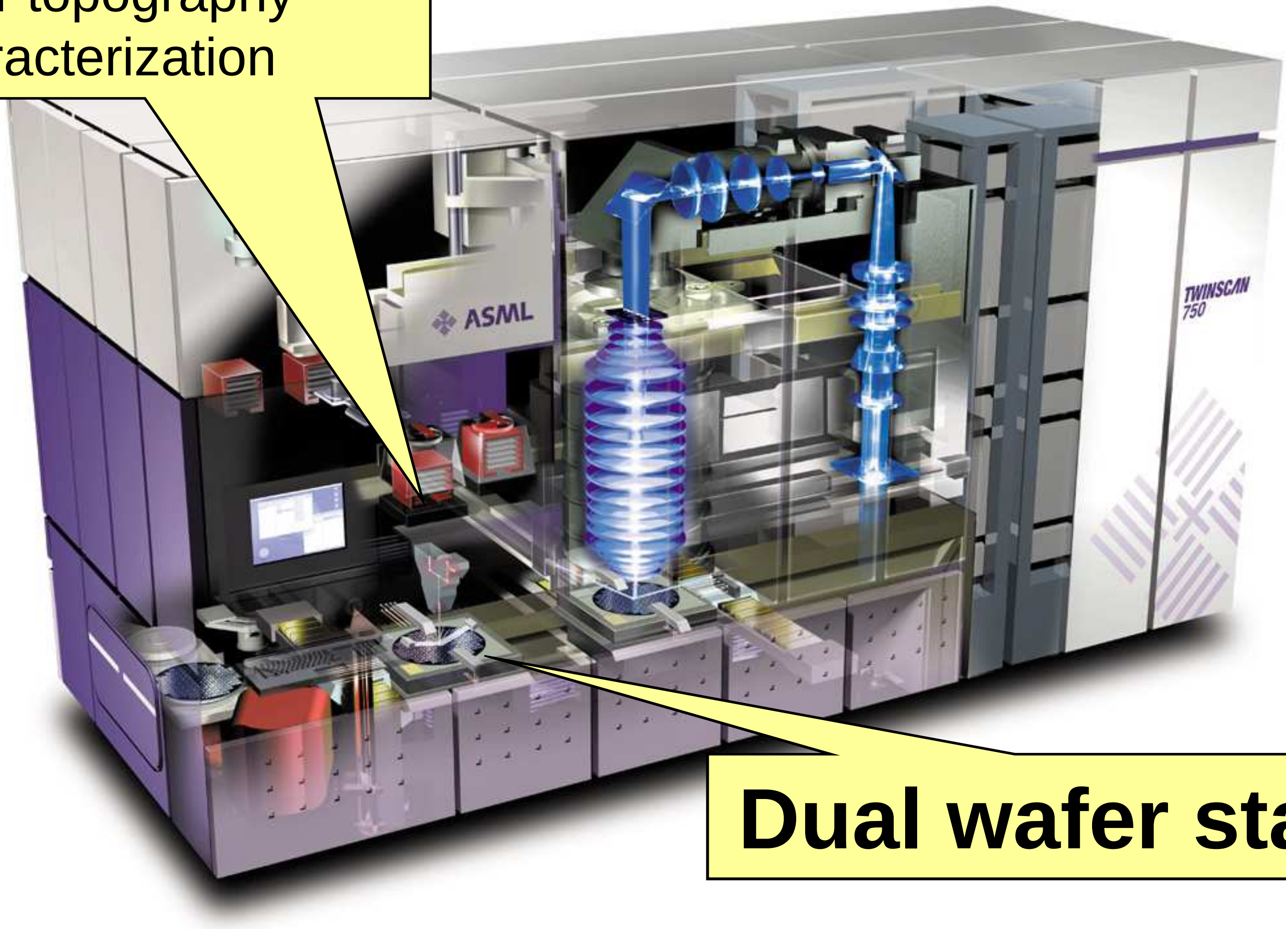
λ = 193 nm, NA = 1.2, res = 45 nm

⇒ Suitable for nanophotonic devices

ASML Wafer Stepper TWINSCAN™:
Deep-UV (DUV) Lithography at 193 nm

Karlsruhe Institute of Technology

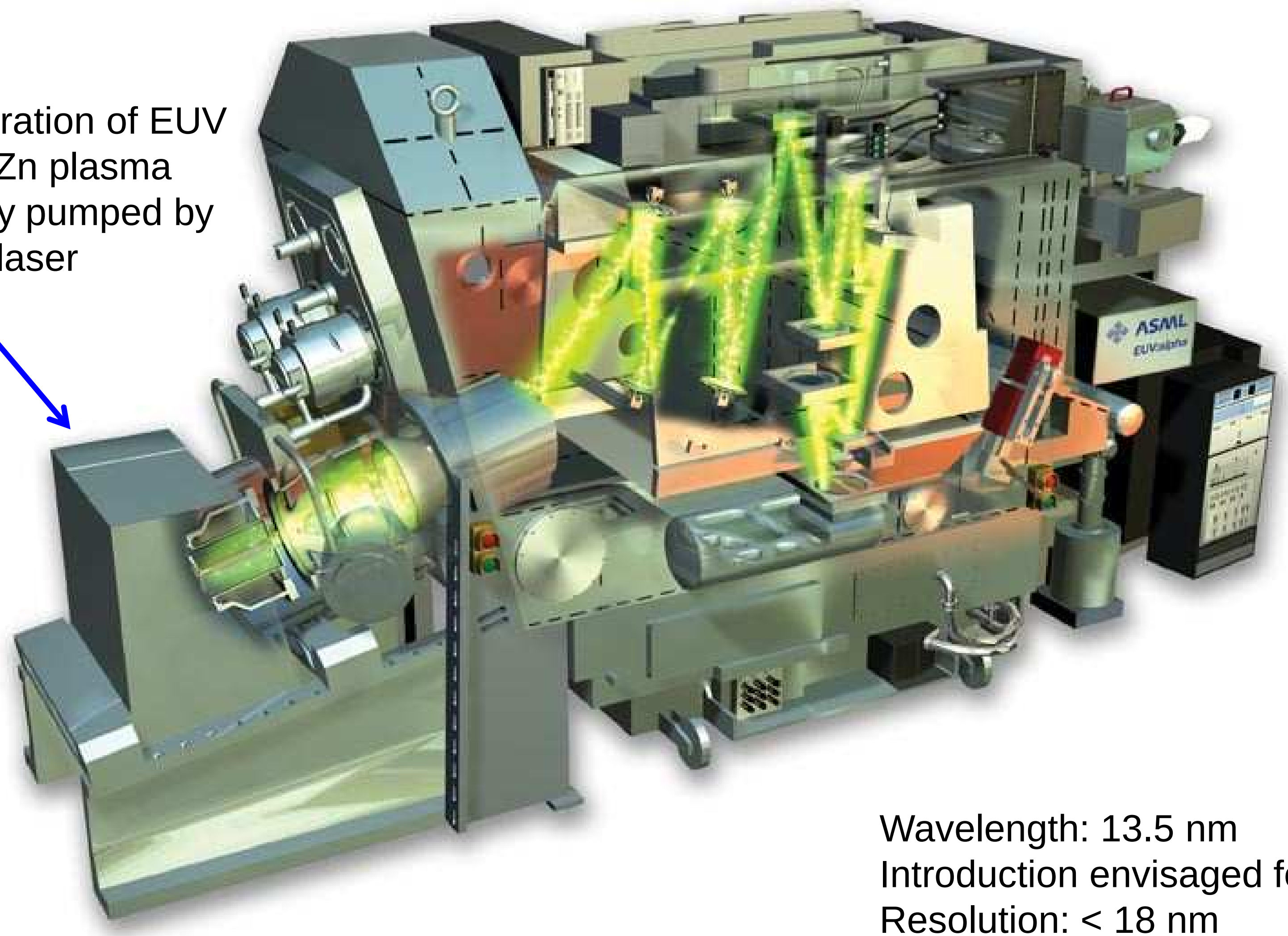
Separate metrology
wafer topography
characterization



Dual wafer stage

Next-generation lithography: Extreme UV (EUV)

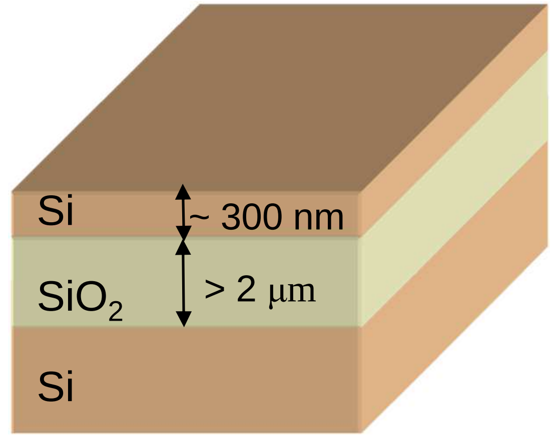
Source: Generation of EUV radiation in a Zn plasma that is optically pumped by a high-power laser



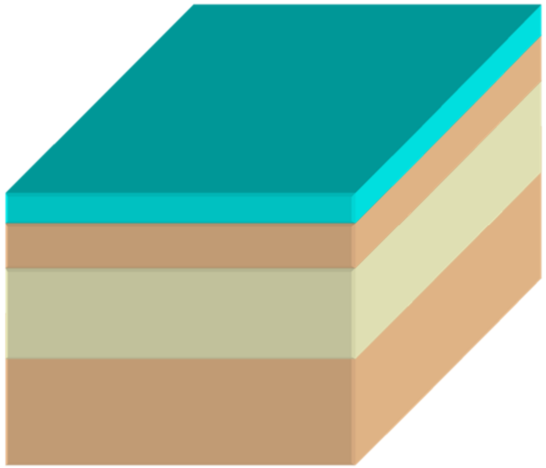
Wavelength: 13.5 nm
Introduction envisaged for 2015
Resolution: < 18 nm

http://www.hardware-infos.com

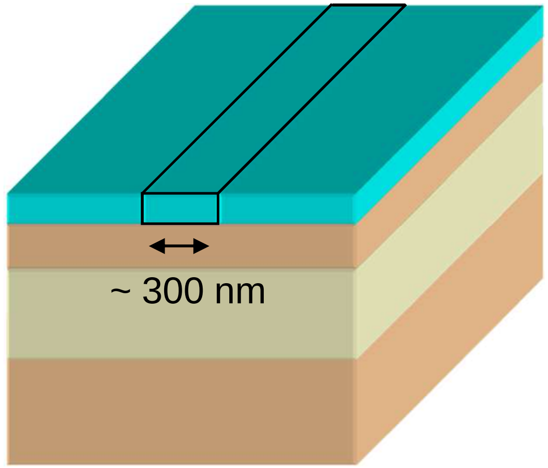
Fabrication of silicon-on-insulator (SOI) waveguides



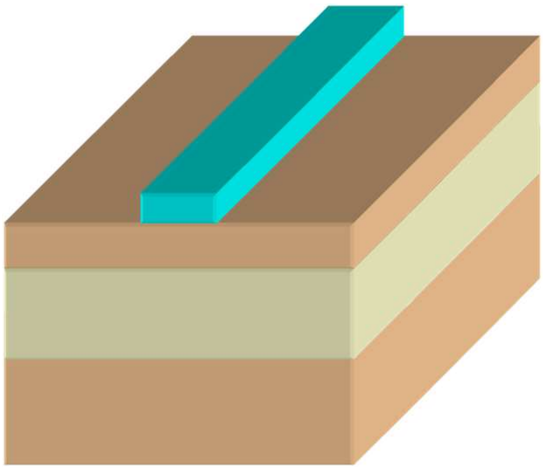
Silicon-on-insulator
(SOI) wafer



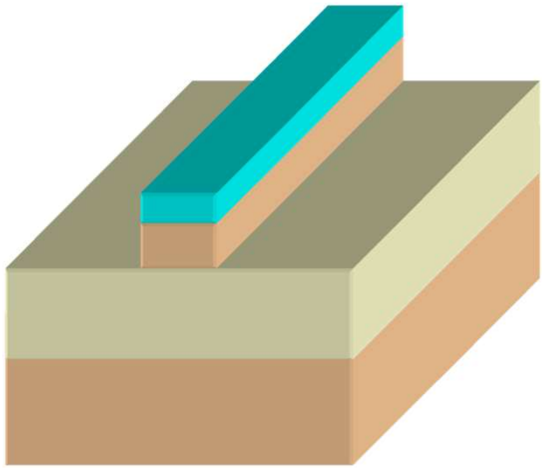
Resist coating



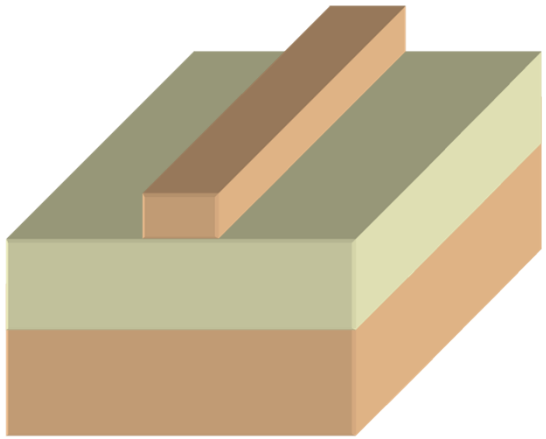
Lithography



Development

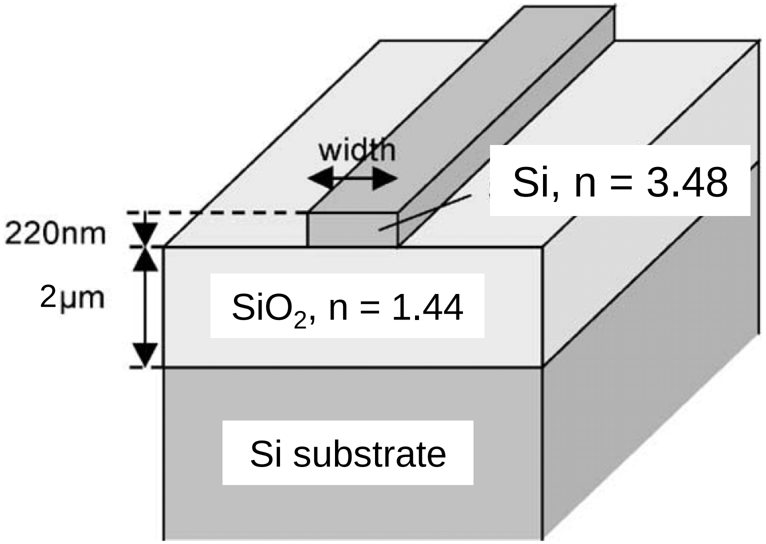


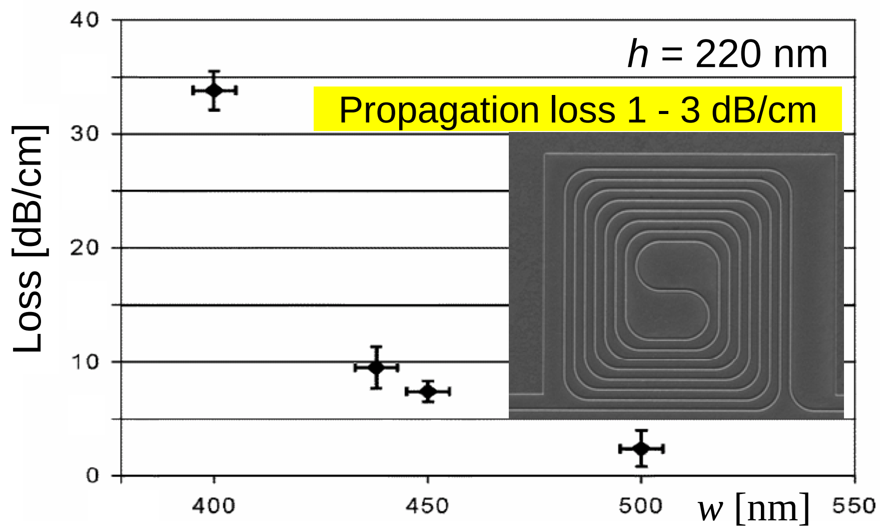
Silicon etching
(e.g., reactive ion etching)

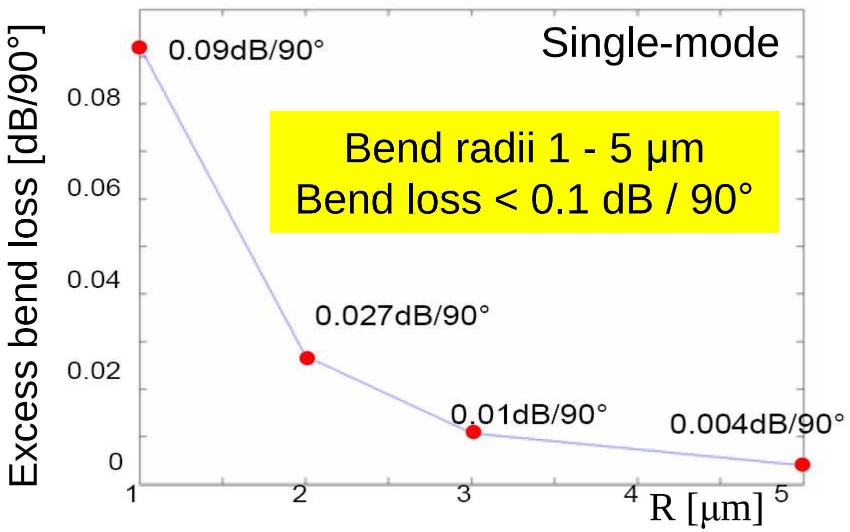


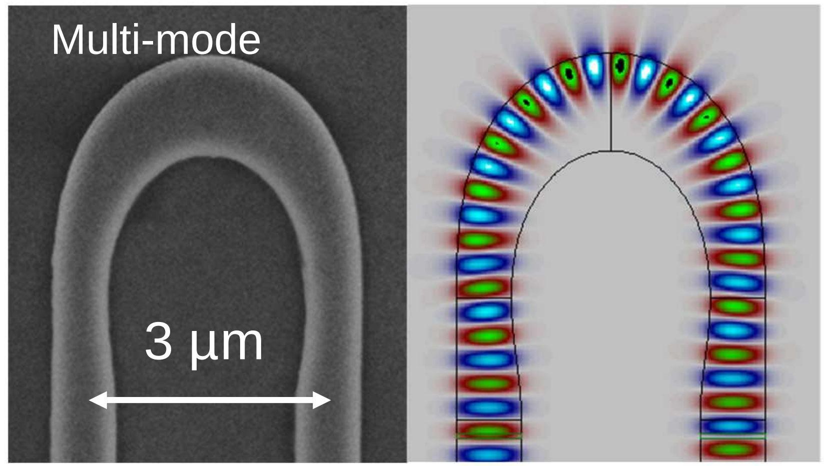
Final photonic wire

SOI waveguides and bends





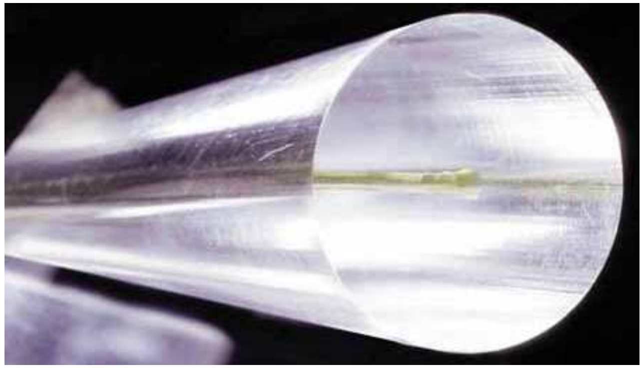


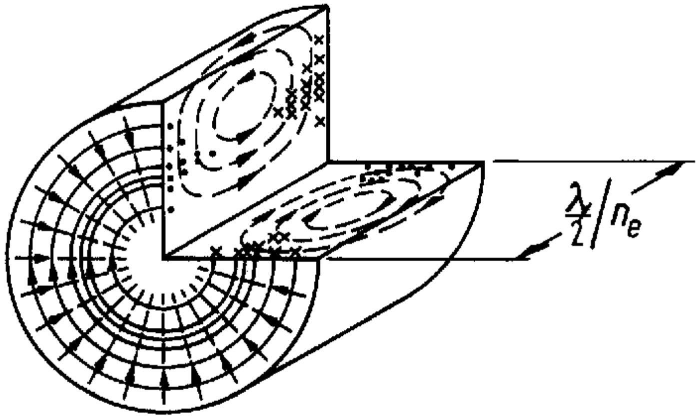


Dumon *et al.*, Photon. Technol. Lett. 16 (2004), 1328-1330

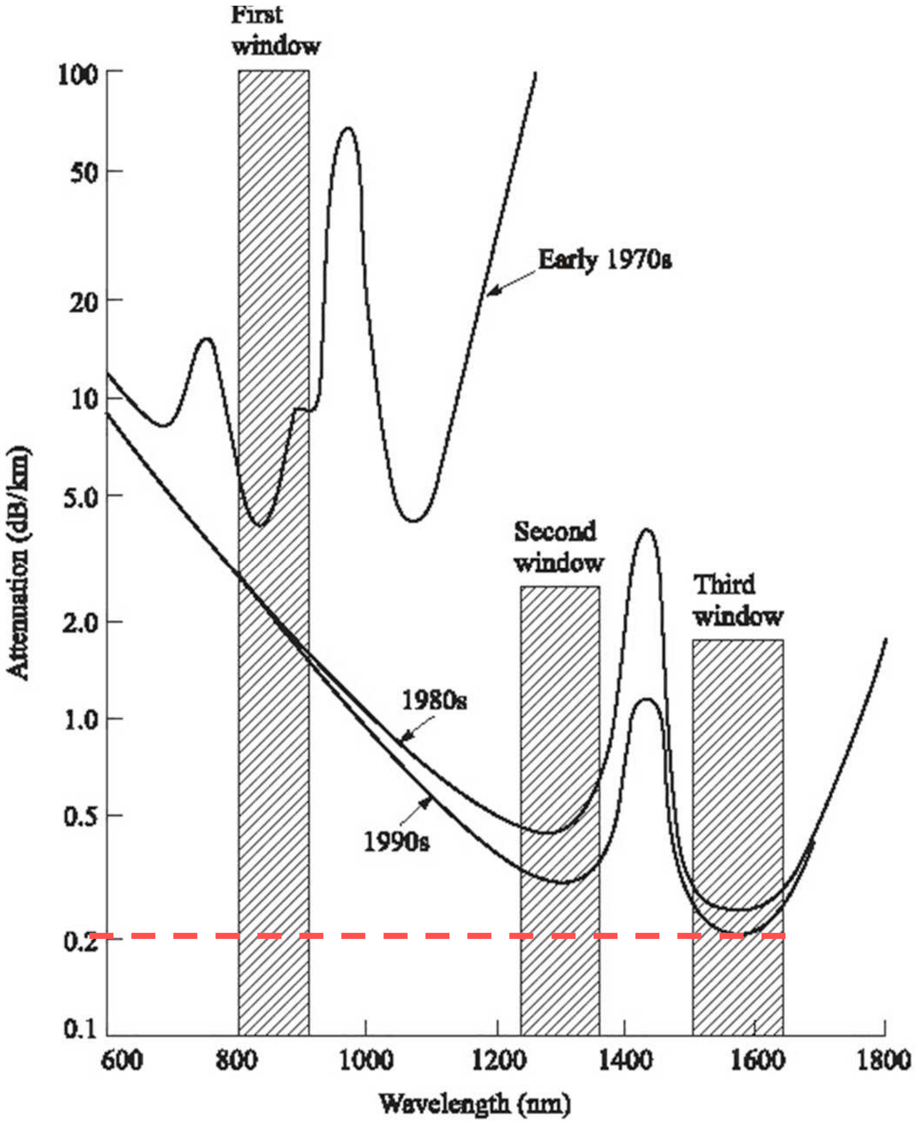
Koos *et al.*, IEEE Photon. Technol. Lett. 19 (2007)

Optical Fibers





Fiber loss and transmission windows



Keiser: Optical Fiber Communications

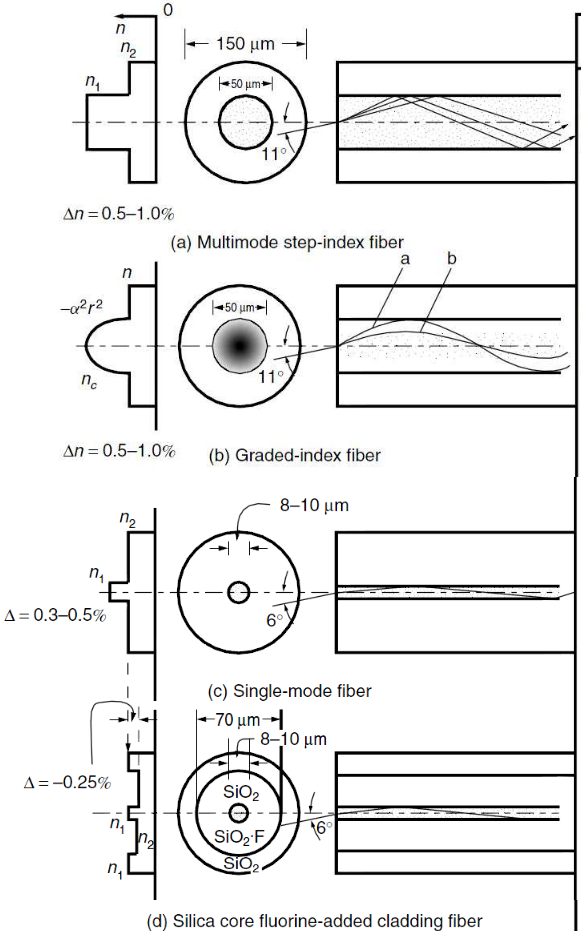
1966: Charles K. Kao showed that the losses of > 1000 dB/km in existing glass was due to impurities and can in principle be reduced to below 20 dB/km.
⇒ Optical fibers as transmission media; Nobel Prize in Physics 2009

1970's: Optical communications in the first transmission window (800 – 900 nm); GaAs-based optical sources and detectors.

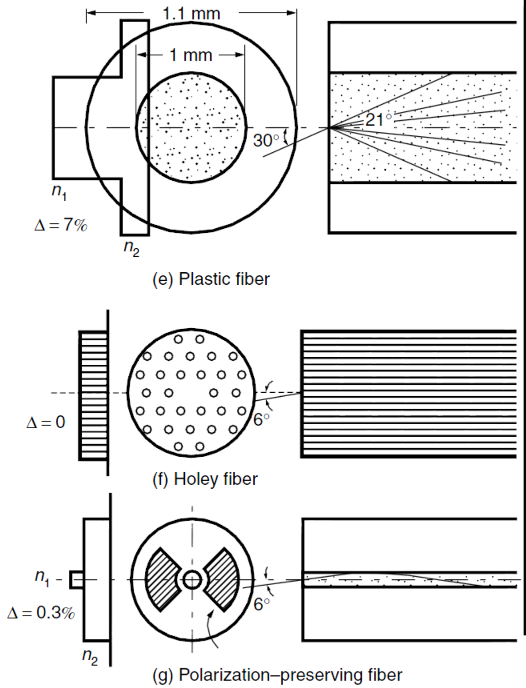
Early 1980's: Further reduction of OH^--ions and metallic ion impurities
⇒ Second-generation fiber-optic transmission at around 1300 nm using InGaAsP lasers

Late 1980's: Third transmission window around 1550 nm. Typical fiber losses of about 0.2 dB/km (Record: 0.154 dB/km in fiber with low-index F-doped cladding); requires control of chromatic dispersion!

Various kinds of optical fibers



Applications	Features	
	Advantage	Disadvantage
Short distance, low bit rate Local area	Easy coupling	Large dispersion
Medium distance, medium bit rate Local area	Easy coupling	Moderate dispersion
Long distance, high bit rate Wide area	Wideband	Difficult coupling
Long distance, high bit rate Wide area Submarine	Immune to H_2 or γ-ray irradiation	Difficult coupling



Applications	Features	
	Advantage	Disadvantage
Extremely short distance Robotic circuits Process control Data link	Easy coupling	High loss Large dispersion
High power, low dispersion	Single material Large core	
Local oscillator, Sensor, Pigtail for integrated optics	No polarization jitter	Higher loss

Iizuka, Elements of Photonics, Vol. 2

Types of fibers:

- Step-index fibers
- Graded-index fibers
- Fibers with non rotation-symmetric index profiles

Recall: Cylindrical coordinates

Koordinatentransformation

$x = \rho \cos\varphi, \quad y = \rho \sin\varphi, \quad z = z$

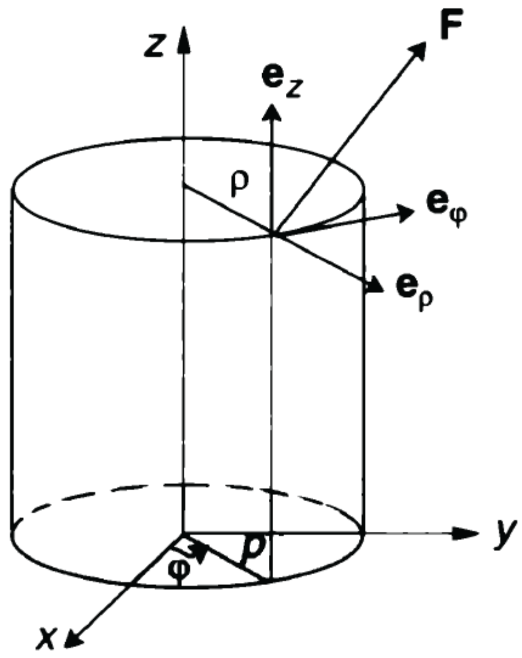
$\rho = \sqrt{x^2 + y^2}, \ \varphi = \tan^{-1} \frac{y}{x}$ (passender Zweig), $z = z$

$h_1 = 1, h_2 = \rho, h_3 = 1$

Beziehung zwischen Basisvektoren

$$\begin{cases} \mathbf{e}_x = \mathbf{e}_\rho \cos\varphi - \mathbf{e}_\varphi \sin\varphi \\ \mathbf{e}_y = \mathbf{e}_\rho \sin\varphi + \mathbf{e}_\varphi \cos\varphi \\ \mathbf{e}_z = \mathbf{e}_z \end{cases}$$

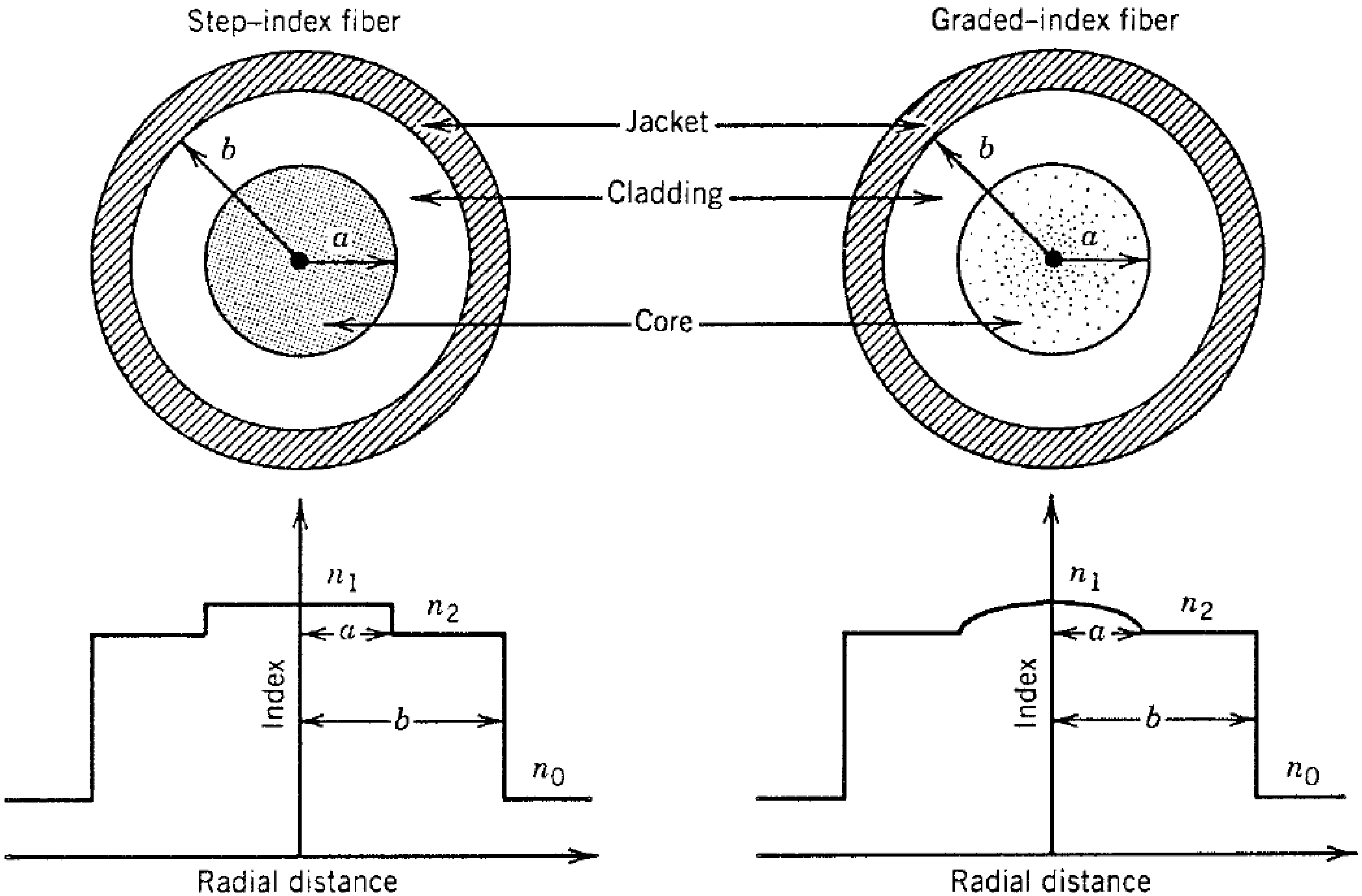
$$\begin{cases} \mathbf{e}_\rho = \mathbf{e}_x \cos\varphi + \mathbf{e}_y \sin\varphi \\ \mathbf{e}_\varphi = -\mathbf{e}_x \sin\varphi + \mathbf{e}_y \cos\varphi \\ \mathbf{e}_z = \mathbf{e}_z \end{cases}$$



Note: We use *r* for the radius rather than ρ …

Rade / Westergren, Mathematische Formeln, Springer

Step-index and graded-index profiles

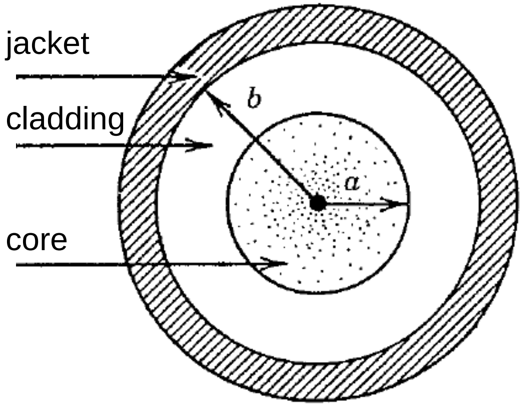


Agrawal, Fiber-Optic Communication Systems

Cylindrical coordinates and refractive index profile

Assumptions:

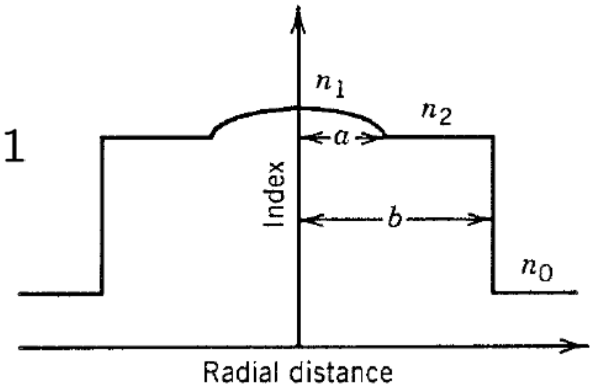
- Fields are confined to the core and the cladding can be assumed to be infinitely thick.
- Later: Low index contrast / small relative index difference $\Delta = \frac{n_1^2 - n_2^2}{2n_1^2} \approx 10^{-3} \ldots 10^{-2}$



Rotation-symmetric refractive index profile:

$$n^2(r) = \begin{cases} n_1^2 \left[1 - 2\Delta\, g\left(\frac{r}{a}\right)\right], & 0 \le r < a \\ n_1^2 \left[1 - 2\Delta\right] = n_2^2, & a \le r < \infty \end{cases}$$

where $g(0) = 0,\ g(1) = 1$

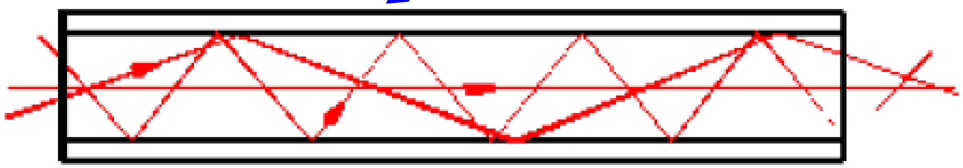


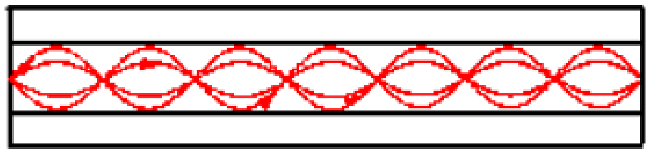
Power-law profiles:

$$g\left(\frac{r}{a}\right) = \left(\frac{r}{a}\right)^q, \qquad 0 \le q < \infty.$$

$q = 2$ Parabolic index profile

$q \to \infty$ Step index profile





Differential operators in cylindrical coordinates

$$\text{grad } u = \nabla u = \frac{\partial u}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial u}{\partial \varphi}\mathbf{e}_\varphi + \frac{du}{dz}\mathbf{e}_z$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z}$$

$$\text{rot } \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z}\right)\mathbf{e}_\rho + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\mathbf{e}_\varphi + \frac{1}{\rho}\left(\frac{\partial(\rho F_\varphi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \varphi}\right)\mathbf{e}_z$$

$$\Delta u = \nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\varphi\varphi} + u_{zz}$$

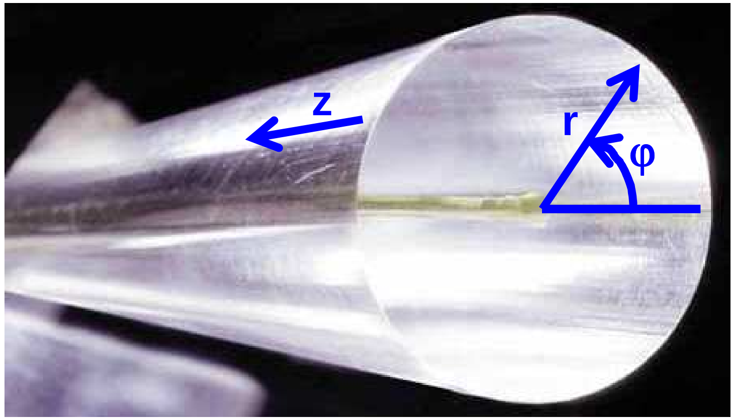
Back to „Bent waveguides“…

Rade / Westergren, Mathematische Formeln, Springer

Maxwell's equations in cylindrical coordinates

Mode ansatz:

$$\underline{\mathbf{E}}(\mathbf{r},t) = \underline{\mathcal{E}}(r,\varphi)\exp\left(\mathrm{j}\left(\omega t - \beta z\right)\right)$$
$$\underline{\mathbf{H}}(\mathbf{r},t) = \underline{\mathcal{H}}(r,\varphi)\exp\left(\mathrm{j}\left(\omega t - \beta z\right)\right)$$
$$\beta = n_e k_0$$



Maxwell's curl equations in polar coordinates:

$$\frac{1}{r}\frac{\partial \underline{\mathcal{E}}_z}{\partial \varphi} + \mathrm{j}\,\beta \underline{\mathcal{E}}_\varphi = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_r$$
$$-\mathrm{j}\,\beta \underline{\mathcal{E}}_r - \frac{\partial \underline{\mathcal{E}}_z}{\partial r} = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_\varphi$$
$$\frac{1}{r}\left(\frac{\partial\left(r\underline{\mathcal{E}}_\varphi\right)}{\partial r} - \frac{\partial \underline{\mathcal{E}}_r}{\partial \varphi}\right) = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_z$$
$$\frac{1}{r}\frac{\partial \underline{\mathcal{H}}_z}{\partial \varphi} + \mathrm{j}\,\beta \underline{\mathcal{H}}_\varphi = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_r$$
$$-\mathrm{j}\,\beta \underline{\mathcal{H}}_r - \frac{\partial \underline{\mathcal{H}}_z}{\partial r} = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_\varphi$$
$$\frac{1}{r}\left(\frac{\partial\left(r\underline{\mathcal{H}}_\varphi\right)}{\partial r} - \frac{\partial \underline{\mathcal{H}}_r}{\partial \varphi}\right) = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_z$$

Transverse field components can be expressed by E_z and H_z :

$$\underline{\mathcal{E}}_r = -\frac{\mathrm{j}}{k_0^2 n^2 - \beta^2}\left(\beta\frac{\partial \underline{\mathcal{E}}_z}{\partial r} + \frac{\omega\mu_0}{r}\frac{\partial \underline{\mathcal{H}}_z}{\partial \varphi}\right)$$
$$\underline{\mathcal{E}}_\varphi = -\frac{\mathrm{j}}{k_0^2 n^2 - \beta^2}\left(\frac{\beta}{r}\frac{\partial \underline{\mathcal{E}}_z}{\partial \varphi} - \omega\mu_0\frac{\partial \underline{\mathcal{H}}_z}{\partial r}\right)$$
$$\underline{H}_r = -\frac{\mathrm{j}}{k_0^2 n^2 - \beta^2}\left(\beta\frac{\partial \underline{\mathcal{H}}_z}{\partial r} - \frac{\omega\epsilon_0 n^2}{r}\frac{\partial \underline{E}_z}{\partial \varphi}\right)$$
$$\underline{H}_\varphi = -\frac{\mathrm{j}}{k_0^2 n^2 - \beta^2}\left(\frac{\beta}{r}\frac{\partial \underline{\mathcal{H}}_z}{\partial \varphi} + \omega\epsilon_0 n^2\frac{\partial \underline{\mathcal{E}}_z}{\partial r}\right)$$

Wave equations for longitudinal components

Components of E- and H-field can be separated

- within homogenous core and cladding regions of step-index fibers
- within weakly inhomogeneous graded-index fibers

Formulate scalar wave equation for longitudinal components in polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \underline{\Psi}(r,\varphi)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \underline{\Psi}(r,\varphi)}{\partial \varphi^2}+\left(k_0^2 n^2(r)-\beta^2\right)\underline{\Psi}(r,\varphi)=0$$

$\underline{\Psi}$ stands for $\underline{\mathcal{E}}_z$ or $\underline{\mathcal{H}}_z$

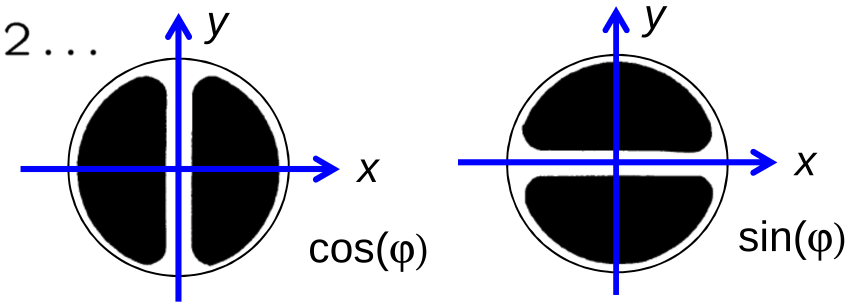
Separation ansatz: $\underline{\Psi}(r,\varphi)=g(r)\;h(\varphi)$

$$\Rightarrow \underbrace{\frac{r}{g(r)}\frac{\partial}{\partial r}\left(r\frac{\partial g(r)}{\partial r}\right)+r^2\left(k_0^2 n^2(r)-\beta^2\right)}_{C_1}+\underbrace{\frac{1}{h(\varphi)}\frac{\partial^2 h(\varphi)}{\partial \varphi^2}}_{-C_1}=0$$

Basic solution for h(φ):

$$h(\varphi)=\begin{cases}\cos(\nu\varphi)\\ \text{or}\\ \sin(\nu\varphi)\end{cases}\quad \text{for } \nu=0,1,2\ldots$$

i.e., modes exist with two different φ-dependencies, “rotated by 90°”:



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Radial field dependence

Differential equation for g(r):

$$r^2 \frac{\partial^2 g(r)}{\partial r^2} + r \frac{\partial g(r)}{\partial r} + \left[\underbrace{\left(k_0^2 n^2 - \beta^2\right)}_{\text{“}\pm a^2\text{”}} r^2 - \underbrace{\nu^2}_{\text{“}p^2\text{”}}\right] g(r) = 0$$

Compare: Bessel differential equations

1. $x^2 y'' + x y' + (a^2 x^2 - p^2) y = 0 \Leftrightarrow y'' + \frac{y'}{x} + \left(a^2 - \frac{p^2}{x^2}\right) y = 0 \Leftrightarrow$

$$\Leftrightarrow \frac{1}{x}(xy')' + \left(a^2 - \frac{p^2}{x^2}\right) y = 0$$

Lösung: $y = AJ_p(ax) + BY_p(ax)$ [Beachte. $|Y_p(x)| \to \infty$ mit $x \to 0+$]

2. $x^2 y'' + x y' - (a^2 x^2 + n^2) y = 0 \Leftrightarrow y'' + \frac{y'}{x} - \left(a^2 + \frac{n^2}{x^2}\right) y = 0 \Leftrightarrow$

$$\Leftrightarrow \frac{1}{x}(xy')' - \left(a^2 + \frac{n^2}{x^2}\right) y = 0$$

Lösung: $y = AI_n(ax) + BK_n(ax)$

Rade / Westergren, Mathematische Formeln, Springer

Radial field dependence

Differential equation for g(r):

$$r^2 \frac{\partial^2 g(r)}{\partial r^2} + r \frac{\partial g(r)}{\partial r} + \left[\left(k_0^2 n^2 - \beta^2\right) r^2 - \nu^2\right] g(r) = 0$$

Distinguish two cases:

1. $k_{1r}^2 = k_0^2 n_1^2 - \beta^2 > 0$: Inside the waveguide core; $0 \leq r \leq a$ and $n = n_1$.

$$\Rightarrow g(r) = C_4\, \mathrm{J}_\nu \left(u \frac{r}{a}\right) + C_5\, \mathrm{Y}_\nu \left(u \frac{r}{a}\right)$$

   where J_ν (Y_ν) denote Bessel functions of the first (second) kind of order ν, and where the transverse phase constant is given by

$$u = a\sqrt{n_1^2 k_0^2 - \beta^2}$$

2. $k_{2r}^{(i)2} = \beta^2 - k_0^2 n_2^2 > 0$: In the cladding, $r > a$ and $n = n_2$.

$$\Rightarrow g(r) = C_6\, \mathrm{I}_\nu \left(u \frac{r}{a}\right) + C_7\, \mathrm{K}_\nu \left(u \frac{r}{a}\right)$$

   where I_ν and K_ν denote modified Bessel functions. The transverse cladding attenuation constant is given by

$$w = a\sqrt{\beta^2 - n_2^2 k_0^2}$$

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„Physically meaningful“ solutions

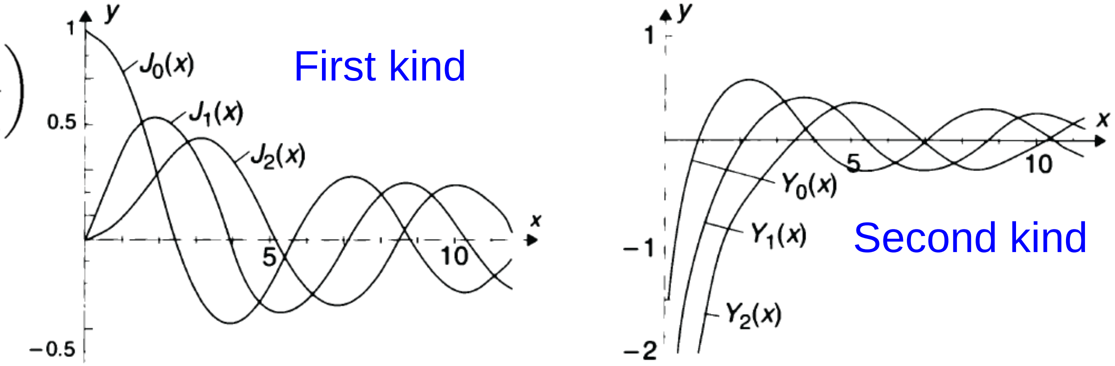
Core: $0 \leq r \leq a$

$$g(r) = C_4 \, \mathrm{J}_\nu \left(u \frac{r}{a} \right) + C_5 \, \mathrm{Y}_\nu \left(u \frac{r}{a} \right)$$

But: $\left| \mathrm{Y}_\nu \left(u \frac{r}{a} \right) \right| \to \infty$ for $r \to 0$

$$\Rightarrow C_5 = 0$$

Bessel-Funktionen



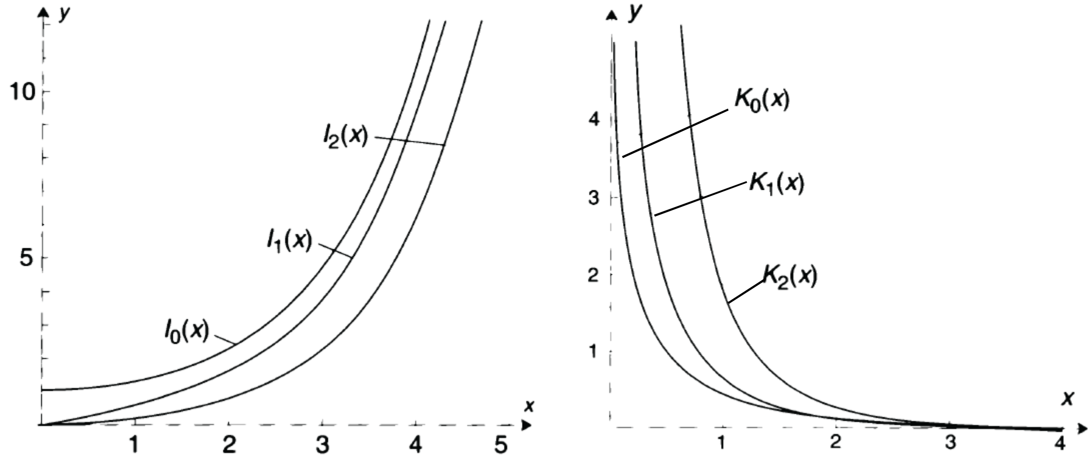
Cladding: $r > a$

$$g(r) = C_6 \, \mathrm{I}_\nu \left(w \frac{r}{a} \right) + C_7 \, \mathrm{K}_\nu \left(w \frac{r}{a} \right)$$

But: $\left| \mathrm{I}_\nu \left(w \frac{r}{a} \right) \right| \to \infty$ for $r \to \infty$

$$\Rightarrow C_6 = 0$$

Modifizierte Bessel-Funktionen



Back to LP modes

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Complete solution

Recombine separated functions:

$$\underline{\Psi}(r,\varphi) = \begin{cases} A\ \mathrm{J}_\nu\left(u\frac{r}{a}\right)\ \cos(\nu\varphi+\psi) & \text{for } 0 \le r \le a \\ A\,\frac{\mathrm{J}_\nu(u)}{\mathrm{K}_\nu(w)}\mathrm{K}_\nu\left(w\frac{r}{a}\right)\ \cos(\nu\varphi+\psi) & \text{for } r > a \end{cases}$$

where $\nu = 0, 1, \ldots$ and $\psi \in \left\{0; \frac{\pi}{2}\right\}$, and where $\underline{\Psi}$ stands for $\underline{\mathcal{E}}_z$ or $\underline{\mathcal{H}}_z$

Recall: $\underline{\mathcal{E}}_r = -\frac{\mathrm{j}}{k_0^2 n^2 - \beta^2}\left(\beta\frac{\partial\underline{\mathcal{E}}_z}{\partial r} + \frac{\omega\mu_0}{r}\frac{\partial\underline{\mathcal{H}}_z}{\partial\varphi}\right)$

$\Rightarrow$ $\frac{\partial\underline{\mathcal{E}}_z}{\partial r}$ and $\frac{\partial\underline{\mathcal{H}}_z}{\partial\varphi}$ and must have the same φ-dependence.

$\Rightarrow$ If $\underline{\mathcal{E}}_z$ has a $\cos(\nu\varphi)$-dependence, $\underline{\mathcal{H}}_z$ must vary like $\sin(\nu\varphi)$ and vice versa.

Complete solution for E_z and H_z-component:

$$\underline{\mathcal{E}}_z(r,\varphi) = \begin{cases} A\ \mathrm{J}_\nu\left(u\frac{r}{a}\right)\ \cos(\nu\varphi+\psi) & \text{for } 0 \le r \le a \\ A\,\frac{\mathrm{J}_\nu(u)}{\mathrm{K}_\nu(w)}\mathrm{K}_\nu\left(w\frac{r}{a}\right)\ \cos(\nu\varphi+\psi) & \text{for } r > a \end{cases}$$

$$\underline{\mathcal{H}}_z(r,\varphi) = \begin{cases} B\ \mathrm{J}_\nu\left(u\frac{r}{a}\right)\ \sin(\nu\varphi+\psi) & \text{for } 0 \le r \le a \\ B\,\frac{\mathrm{J}_\nu(u)}{\mathrm{K}_\nu(w)}\mathrm{K}_\nu\left(w\frac{r}{a}\right)\ \sin(\nu\varphi+\psi) & \text{for } r > a \end{cases}$$

where $\nu = 0, 1, \ldots$ and $\psi \in \left\{0; \frac{\pi}{2}\right\}$

Recurrence relations for Bessel functions

Für jede *Zylinderfunktion* $C_p(x) = J_p(x),\ Y_p(x),\ H_p^{(1)}(x)$ oder $H_p^{(2)}(x)$ gilt

$$C_{p-1}(x) + C_{p+1}(x) = \frac{2p}{x}\, C_p(x) \qquad C_{p-1}(x) - C_{p+1}(x) = 2C_p'(x)$$

$$xC_p'(x) = pC_p(x) - xC_{p+1}(x) = xC_{p-1}(x) - pC_p(x)$$

$$\frac{d}{dx}\,\{x^p C_p(x)\} = x^p C_{p-1}(x) \qquad \frac{d}{dx}\,\{x^{-p} C_p(x)\} = -x^{-p} C_{p+1}(x)$$

Speziell. $J_0'(x) = -J_1(x),\ Y_0'(x) = -Y_1(x)$

$$\int C_n^2(x)\, x\, dx = \frac{1}{2}\, x^2 [C_n'(x)]^2 + \frac{1}{2}\,(x^2 - n^2) C_n^2(x)$$

$$\int x^{1+n} C_n(x) dx = x^{1+n} C_{n+1}(x) = -x^{1-n}\left[C_n'(x) - \frac{n}{x}\, C_n(x)\right]$$

$$\int x^{1-n} C_n(x) dx = -x^{1-n} C_{n-1}(x) = -x^{1-n}\left[C_n'(x) + \frac{n}{x}\, C_n(x)\right]$$

$$\int x^n C_0(x) dx = x^n C_1(x) + (n-1) x^{n-1} C_0(x) - (n-1)^2 \int x^{n-2} C_0(x) dx$$

$$\int C_n(\alpha x) C_n(\beta x)\, x\, dx = \frac{x[\alpha C_n(\beta x) C_n'(\alpha x) - \beta C_n(\alpha x) C_n'(\beta x)]}{\beta^2 - \alpha^2}$$

$$\int C_n^2(\alpha x)\, x\, dx = \frac{x^2}{2}\left[C_n'(\alpha x)^2 + \left(1 - \frac{n^2}{\alpha^2 x^2}\right) C_n(\alpha x)^2\right]$$

$$I_{n+1}(x) = I_{n-1}(x) - \frac{2n}{x}\, I_n(x) = 2I_n'(x) - I_{n-1}(x)$$

$$K_{n+1}(x) = K_{n-1}(x) + \frac{2n}{x}\, K_n(x) = -2K_n'(x) - K_{n-1}(x)$$

Back to LP modes …

TE modes

For $\nu = 0$, $\psi = \pi/2$, we have: $\underline{\mathcal{E}}_z(r,\varphi) = 0$

$$\underline{\mathcal{H}}_z(r,\varphi) = \begin{cases} B\ \mathrm{J}_0\left(u\frac{r}{a}\right) & \text{for } 0 \le r \le a \\ B\ \frac{\mathrm{J}_0(u)}{\mathrm{K}_0(w)}\mathrm{K}_0\left(w\frac{r}{a}\right) & \text{for } r > a \end{cases}$$

Recall:

$$\underline{\mathcal{E}}_r = -\frac{\mathrm{j}}{k_0^2 n^2 - \beta^2}\left(\beta\frac{\partial\underline{\mathcal{E}}_z}{\partial r} + \frac{\omega\mu_0}{r}\frac{\partial\underline{\mathcal{H}}_z}{\partial\varphi}\right)$$

$$\underline{\mathcal{E}}_\varphi = -\frac{\mathrm{j}}{k_0^2 n^2 - \beta^2}\left(\frac{\beta}{r}\frac{\partial\underline{\mathcal{E}}_z}{\partial\varphi} - \omega\mu_0\frac{\partial\underline{\mathcal{H}}_z}{\partial r}\right)$$

The other E-field components can be derived using recurrence relations for Bessel functions:

$$\underline{\mathcal{E}}_r = 0$$

$$\underline{\mathcal{E}}_\varphi = \mathrm{j}\,\omega\mu_0 \begin{cases} -\frac{Ba}{u}\ \mathrm{J}_1\left(u\frac{r}{a}\right) & \text{for } 0 \le r \le a \\ \frac{Ba}{w}\frac{\mathrm{J}_0(u)}{\mathrm{K}_0(w)}\mathrm{K}_1\left(w\frac{r}{a}\right) & \text{for } r > a \end{cases}$$

Continuity of E_φ at r = a yields the dispersion equation for TE modes:

$$\frac{\mathrm{J}_1(u)}{u\,\mathrm{J}_0(u)} = -\frac{\mathrm{K}_1(w)}{w\,\mathrm{K}_0(w)} \qquad \text{where} \qquad u^2 + w^2 = V^2$$

The same relation is reproduced by the continuity requirements of H_r

TM modes and hybrid modes

For $\nu = 0$, $\psi = 0$, we have:

$$\mathcal{E}_z(r,\varphi) = \begin{cases} A\,\mathrm{J}_0\left(u\frac{r}{a}\right) & \text{for } 0 \le r \le a \\ A\,\frac{\mathrm{J}_0(u)}{\mathrm{K}_0(w)}\,\mathrm{K}_0\left(u\frac{r}{a}\right) & \text{for } r > a \end{cases}$$

$$\mathcal{H}_z(r,\varphi) = 0$$

Continuity of E_ρ and H_φ at r = a yields the dispersion equation for TM modes:

$$\frac{\mathrm{J}_1(u)}{u\,\mathrm{J}_0(u)} = -\left(\frac{n_2}{n_1}\right)^2 \frac{\mathrm{K}_1(w)}{w\,\mathrm{K}_0(w)} \qquad \text{where} \qquad u^2 + w^2 = V^2$$

Similarly, a dispersion equation can be derived for the general case where neither H_z nor E_z are zero:

$$\left[\frac{\mathrm{J}_\nu'(u)}{u\,\mathrm{J}_\nu(u)} + \frac{\mathrm{K}_\nu'(w)}{w\,\mathrm{K}_\nu(w)}\right]\left[\frac{\mathrm{J}_\nu'(u)}{u\,\mathrm{J}_\nu(u)} + \left(\frac{n_2}{n_1}\right)^2 \frac{\mathrm{K}_\nu'(w)}{w\,\mathrm{K}_\nu(w)}\right] = \nu^2\left[\frac{1}{u^2} + \frac{1}{w^2}\right]\left[\frac{1}{u^2} + \left(\frac{n_2}{n_1}\right)^2 \frac{1}{w^2}\right]$$

Numerical solution of these equations yields the dispersion relation in its usual form …

Dispersion relations of the step-index fiber

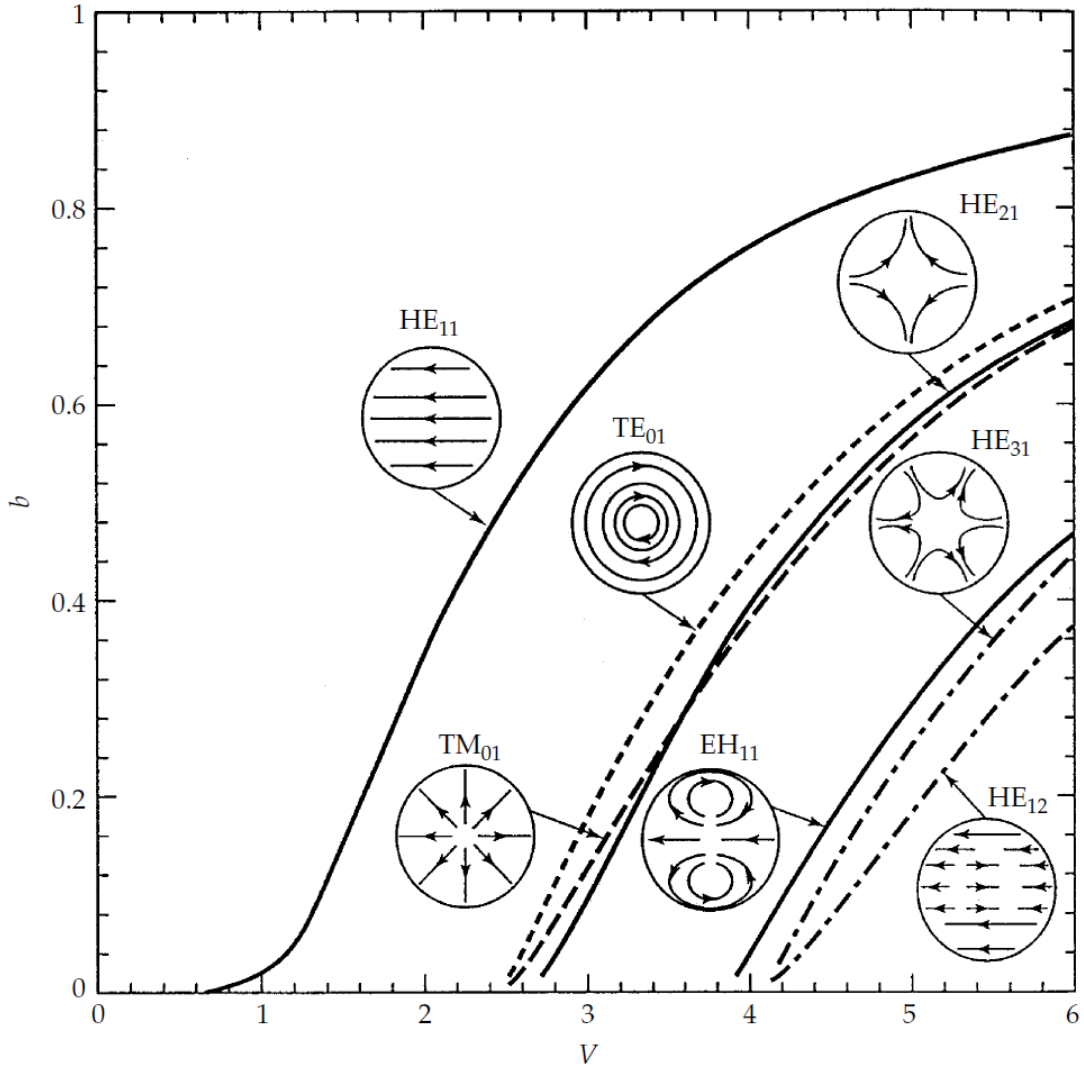


Figure 9.3 Dispersion of a fiber with $\Delta = 0.2$ and electric field lines of various modes.

Chen, Foundations for Guided-Wave Optics, Wiley

- TE- and TM modes
  TE: $E_r = E_z = 0$; $E_\varphi \neq 0$
  TM: $H_r = H_z = 0$; $H_\varphi \neq 0$
- Nomenclature of hybrid modes:

  dominant longitudinal H-field component → $\mathrm{EH}_{\nu\mu}$

  dominant longitudinal E-field component → $\mathrm{HE}_{\nu\mu}$

  Azimuthal mode index (ν), Radial mode index (μ)
- Fundamental mode of step-index fibers: HE_{11}-mode
- Single-mode condition: V < 2.405

  (first zero of $J_0(u)$)
- Groups of modes with similar dispersion characteristics, e.g., TE_{01}, TM_{01} and HE_{21}; these modes will lateron be merged to so-called linearly polarized ($LP\nu\mu$) modes for weakly guiding waveguides.

TE and TM: Mode designation and fields

TE$_{\nu\mu}$ modes:

- Exist only for ν = 0 => rotational symmetry!
- μ denotes the number of maxima of E_φ along r (including the one at r = 0, if existent)
- Field components:

$$\underline{\mathcal{E}}_r = 0;\ \underline{\mathcal{E}}_\varphi \neq 0;\ \underline{\mathcal{E}}_z = 0$$

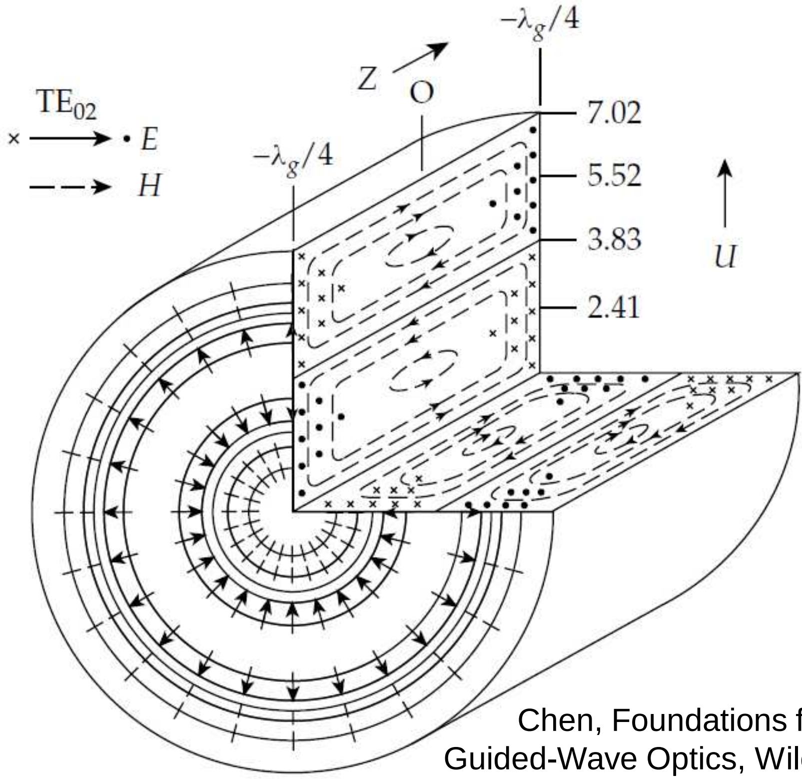
$$\underline{\mathcal{H}}_r \neq 0;\ \underline{\mathcal{H}}_\varphi = 0;\ \underline{\mathcal{H}}_z \neq 0$$

TM$_{\nu\mu}$ modes:

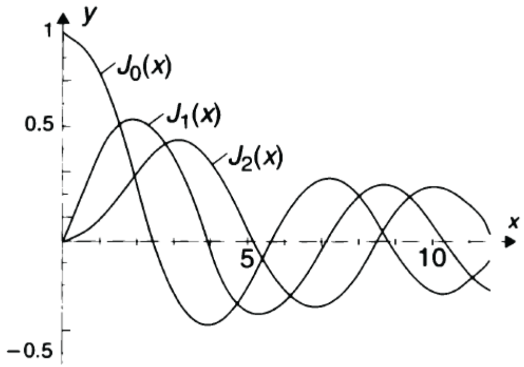
- Exist only for ν = 0 => rotational symmetry!
- μ denotes the number of maxima of H_φ along r (including the one at r = 0, if existent)
- Field components:

$$\underline{\mathcal{E}}_r \neq 0;\ \underline{\mathcal{E}}_\varphi = 0;\ \underline{\mathcal{E}}_z \neq 0$$

$$\underline{\mathcal{H}}_r = 0;\ \underline{\mathcal{H}}_\varphi \neq 0;\ \underline{\mathcal{H}}_z = 0$$



Chen, Foundations for Guided-Wave Optics, Wiley

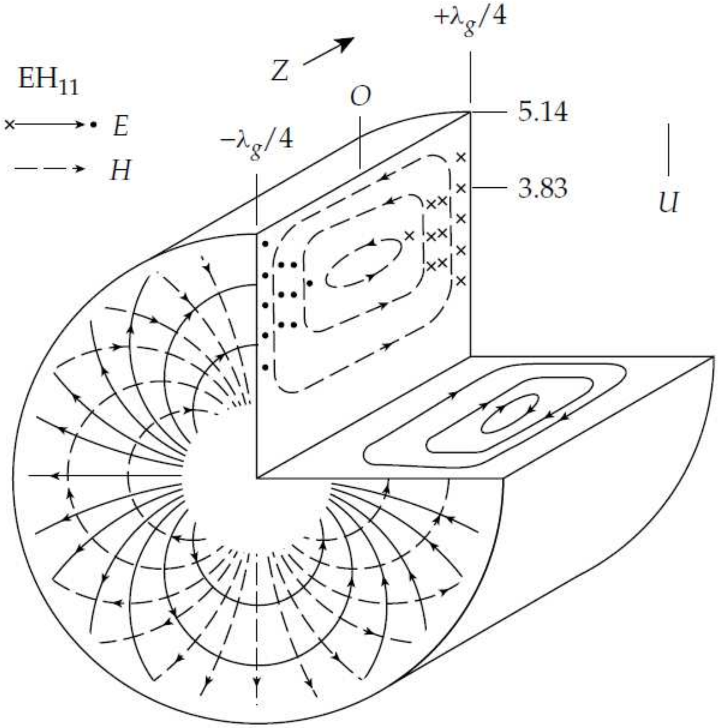


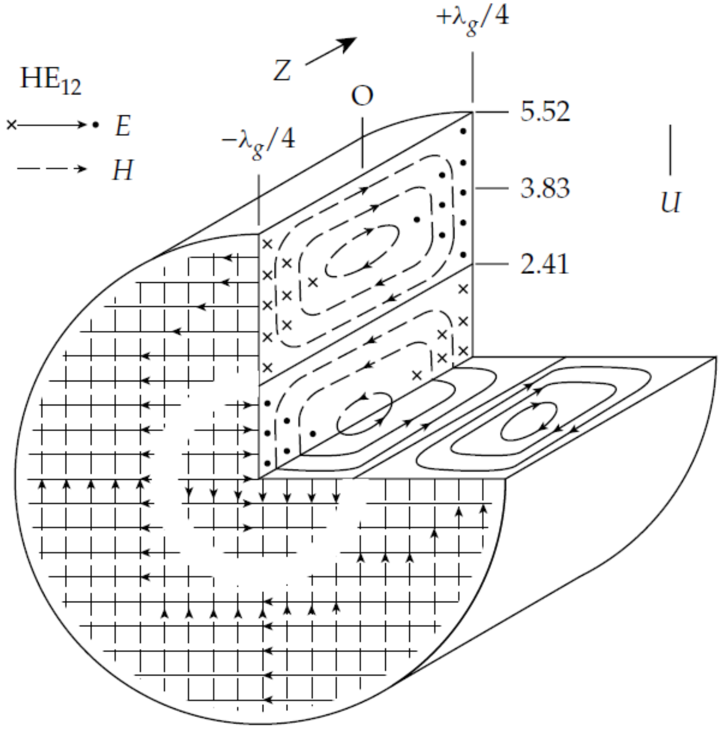
Hybrid mode designation and fields

EH$_{\nu\mu}$- and HE$_{\nu\mu}$-modes:

- Exist for $\nu \geq 1$; ν denotes the azimuthal symmetry
- Each mode is two-fold degenerate: sin(νφ) and cos(νφ)-dependence
- (Conventional, somewhat arbitrary) nomenclature of EH$_{\nu\mu}$ and HE$_{\nu\mu}$ –modes according to the field with dominant z-component:

H_z dominates => EH$_{\nu\mu}$ –mode; E_z dominates => HE$_{\nu\mu}$ –mode



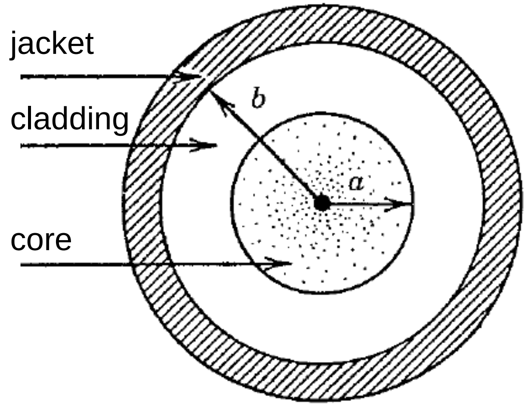


Chen, Foundations for Guided-Wave Optics, Wiley

Optical fibers: Cylindrical coordinates and refractive index profile

Assumptions:

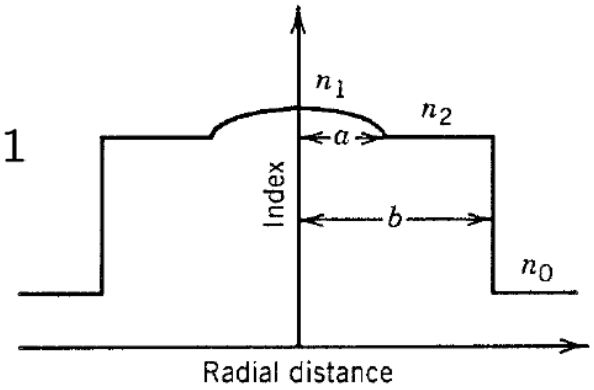
- Fields are confined to the core, i.e., the cladding can be assumed to be infinitely thick.
- Later: Low index contrast / small relative index difference $$\Delta = \frac{n_1^2 - n_2^2}{2n_1^2} \approx 10^{-3} \ldots 10^{-2}$$



Rotation-symmetric refractive index profile:

$$n^2(r) = \begin{cases} n_1^2 \left[1 - 2\Delta\, g\left(\frac{r}{a}\right)\right], & 0 \le r < a \\ n_1^2\left[1 - 2\Delta\right] = n_2^2, & a \le r < \infty \end{cases}$$

where $g(0) = 0,\ g(1) = 1$

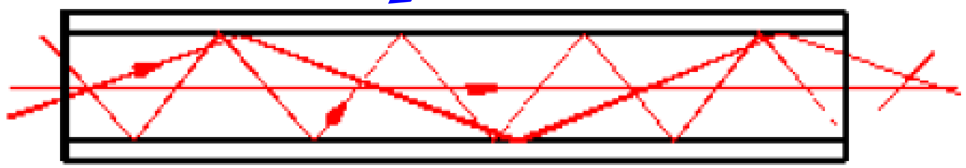


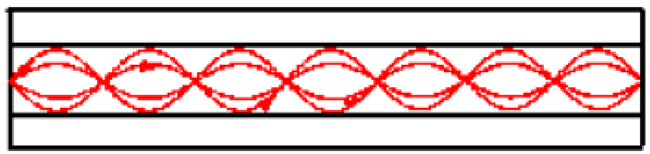
Power-law profiles:

$$g\left(\frac{r}{a}\right) = \left(\frac{r}{a}\right)^q, \qquad 0 \le q < \infty.$$

$q = 2$ Parabolic index profile

$q \to \infty$ Step index profile





Step-index fiber: Wave equations for longitudinal components

Components of E- and H-field can be separated
- within homogenous core and cladding regions of step-index fibers
- within weakly inhomogeneous graded-index fibers

Formulate scalar wave equation for longitudinal components in polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\underline{\Psi}(r,\varphi)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\underline{\Psi}(r,\varphi)}{\partial\varphi^2}+\left(k_0^2n^2(r)-\beta^2\right)\underline{\Psi}(r,\varphi)=0$$

$\underline{\Psi}$ stands for $\underline{\mathcal{E}}_z$ or $\underline{\mathcal{H}}_z$

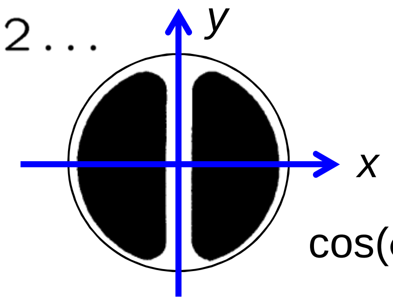
Separation ansatz: $\underline{\Psi}(r,\varphi)=g(r)\,h(\varphi)$

$$\Rightarrow\underbrace{\frac{r}{g(r)}\frac{\partial}{\partial r}\left(r\frac{\partial g(r)}{\partial r}\right)+r^2\left(k_0^2n^2(r)-\beta^2\right)}_{C_1}+\underbrace{\frac{1}{h(\varphi)}\frac{\partial^2 h(\varphi)}{\partial\varphi^2}}_{-C_1}=0$$

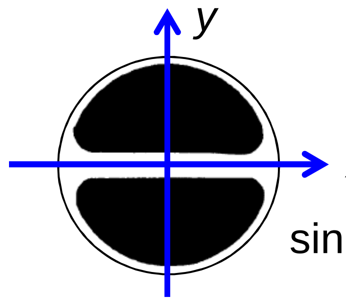
Basic solution for h(φ):

$$h(\varphi)=\begin{cases}\cos(\nu\varphi)\\ \text{or}\\ \sin(\nu\varphi)\end{cases}\quad\text{for } \nu=0,1,2\ldots$$

i.e., modes exist with two different φ-dependencies, “rotated by 90°”:



cos(φ)



sin(φ)

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Dependence on *r*: „Physically meaningful“ solutions

Karlsruhe Institute of Technology

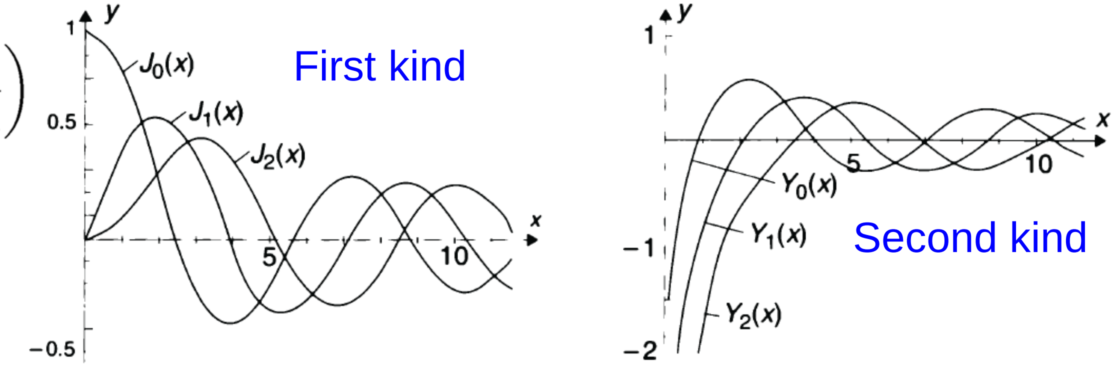
Core: $0 \leq r \leq a$

$$g(r) = C_4\,\mathrm{J}_\nu\left(u\frac{r}{a}\right) + C_5\,\mathrm{Y}_\nu\left(u\frac{r}{a}\right)$$

But: $\left|\mathrm{Y}_\nu\left(u\frac{r}{a}\right)\right| \to \infty$ for $r \to 0$

$$\Rightarrow C_5 = 0$$

Bessel-Funktionen



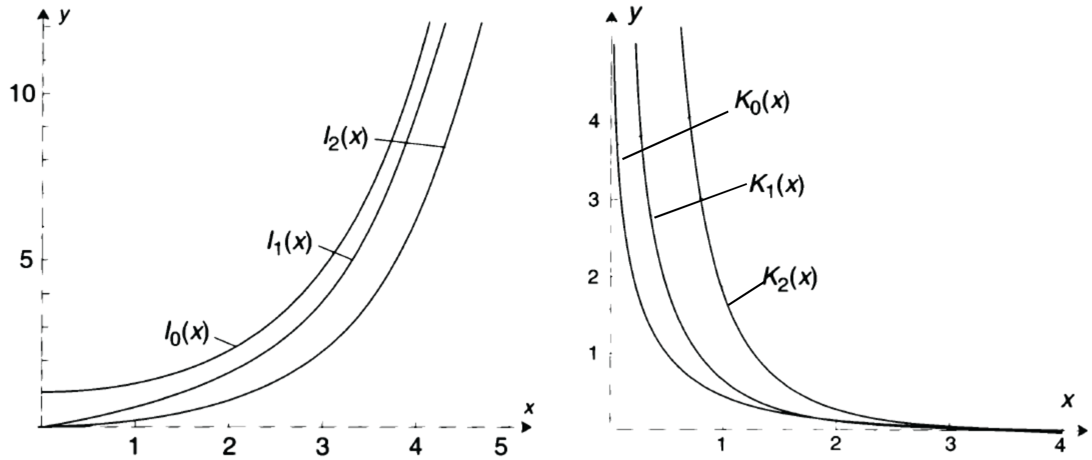
Cladding: $r > a$

$$g(r) = C_6\,\mathrm{I}_\nu\left(w\frac{r}{a}\right) + C_7\,\mathrm{K}_\nu\left(w\frac{r}{a}\right)$$

But: $\left|\mathrm{I}_\nu\left(w\frac{r}{a}\right)\right| \to \infty$ for $r \to \infty$

$$\Rightarrow C_6 = 0$$

Modifizierte Bessel-Funktionen



Back to LP modes

Rade / Westergren, Mathematische Formeln, Springer

Complete solution

Recombine separated functions:

$$\underline{\Psi}(r,\varphi) = \begin{cases} A\ \mathrm{J}_\nu\left(u\frac{r}{a}\right)\ \cos(\nu\varphi+\psi) & \text{for } 0 \le r \le a \\ A\,\frac{\mathrm{J}_\nu(u)}{\mathrm{K}_\nu(w)}\,\mathrm{K}_\nu\left(w\frac{r}{a}\right)\ \cos(\nu\varphi+\psi) & \text{for } r > a \end{cases}$$

where $\nu = 0, 1, \ldots$ and $\psi \in \left\{0; \frac{\pi}{2}\right\}$, and where $\underline{\Psi}$ stands for $\underline{\mathcal{E}}_z$ or $\underline{\mathcal{H}}_z$

Recall: $\underline{\mathcal{E}}_r = -\dfrac{\mathrm{j}}{k_0^2 n^2 - \beta^2}\left(\beta\dfrac{\partial\underline{\mathcal{E}}_z}{\partial r} + \dfrac{\omega\mu_0}{r}\dfrac{\partial\underline{\mathcal{H}}_z}{\partial\varphi}\right)$

$\Rightarrow$ $\frac{\partial\underline{\mathcal{E}}_z}{\partial r}$ and $\frac{\partial\underline{\mathcal{H}}_z}{\partial\varphi}$ and must have the same φ-dependence.

$\Rightarrow$ If $\underline{\mathcal{E}}_z$ has a $\cos(\nu\varphi)$-dependence, $\underline{\mathcal{H}}_z$ must vary like $\sin(\nu\varphi)$ and vice versa.

Complete solution for E_z and H_z-component:

$$\underline{\mathcal{E}}_z(r,\varphi) = \begin{cases} A\ \mathrm{J}_\nu\left(u\frac{r}{a}\right)\ \cos(\nu\varphi+\psi) & \text{for } 0 \le r \le a \\ A\,\frac{\mathrm{J}_\nu(u)}{\mathrm{K}_\nu(w)}\,\mathrm{K}_\nu\left(w\frac{r}{a}\right)\ \cos(\nu\varphi+\psi) & \text{for } r > a \end{cases}$$

$$\underline{\mathcal{H}}_z(r,\varphi) = \begin{cases} B\ \mathrm{J}_\nu\left(u\frac{r}{a}\right)\ \sin(\nu\varphi+\psi) & \text{for } 0 \le r \le a \\ B\,\frac{\mathrm{J}_\nu(u)}{\mathrm{K}_\nu(w)}\,\mathrm{K}_\nu\left(w\frac{r}{a}\right)\ \sin(\nu\varphi+\psi) & \text{for } r > a \end{cases}$$

where $\nu = 0, 1, \ldots$ and $\psi \in \left\{0; \frac{\pi}{2}\right\}$

Dispersion relations of the step-index fiber

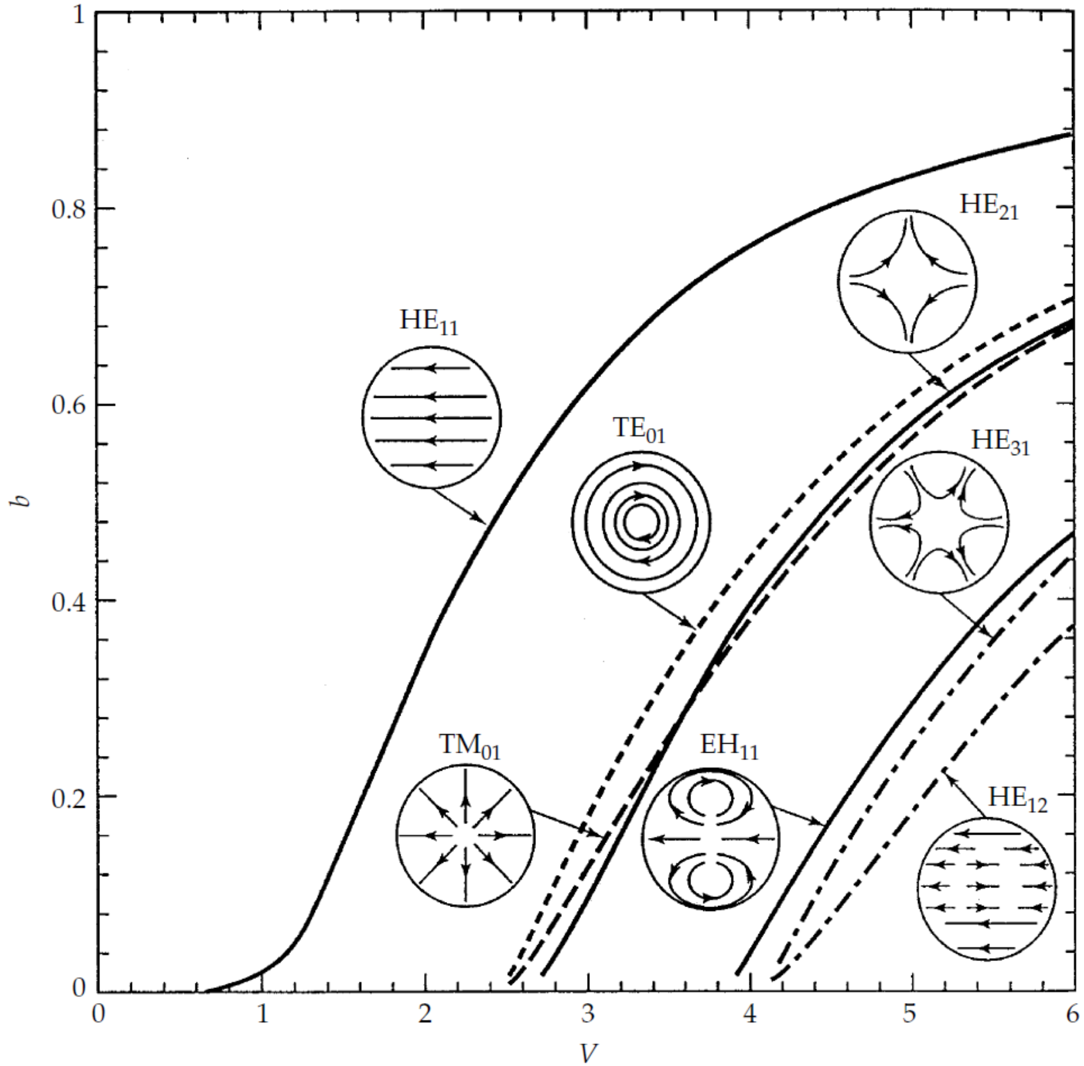


Figure 9.3 Dispersion of a fiber with $\Delta = 0.2$ and electric field lines of various modes.

Chen, Foundations for Guided-Wave Optics, Wiley

- TE- and TM modes
  TE: $E_r = E_z = 0$; $E_\varphi \neq 0$
  TM: $H_r = H_z = 0$; $H_\varphi \neq 0$
- Nomenclature of hybrid modes:

  dominant longitudinal H-field component → $\mathrm{EH}_{\nu\mu}$ ← dominant longitudinal E-field component: $\mathrm{HE}_{\nu\mu}$

  Azimuthal mode index (ν), Radial mode index (μ)
- Fundamental mode of step-index fibers: HE_{11}-mode
- Single-mode condition: V < 2.405

  (first zero of $J_0(u)$)
- Groups of modes with similar dispersion characteristics, e.g., TE_{01}, TM_{01} and HE_{21}; these modes will lateron be merged to so-called linearly polarized (LP$\nu\mu$) modes for weakly guiding waveguides.

Linearly polarized (LP) modes

In weakly guiding fibers ($\Delta \to 0$, $n_1^2/n_2^2 \approx 1$), one transverse Cartesian electric field component is usually much stronger than the other electric field components. Without loss of generality, we assume that the $\underline{\mathcal{E}}_x$-component of the modal field dominates whereas $\underline{\mathcal{E}}_y$ vanishes, $|\underline{\mathcal{E}}_x| \gg |\underline{\mathcal{E}}_z|$ and $|\underline{\mathcal{E}}_y| = 0$. This is in analogy to Marcatili's treatment of a rectangular waveguide. Following this approach, an alternative set of modes, the so-called *linearly polarized (LP)* modes can be derived.

Wave equation for dominant electric field component:

$$\nabla^2 \underline{\mathcal{E}}_x + \left(k_0^2 n^2 - \beta^2\right) \underline{\mathcal{E}}_x = 0$$

Investigate spatial dependence of E_x in polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \underline{\Psi}(r,\varphi)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \underline{\Psi}(r,\varphi)}{\partial \varphi^2} + \left(k_0^2 n^2 - \beta^2\right) \underline{\Psi}(r,\varphi) = 0,$$

where $\underline{\Psi}(r,\varphi)$ represents the $\mathcal{E}_x$-component of the modal field.

Note: Ψ (r,φ) must be "approximately continuous" at r = a!

Linearly polarized (LP) modes

Solution ansatz (in analogy to derivation of hybrid modes!):

$$\underline{\Psi}(r,\varphi) = \begin{cases} A\,\mathrm{J}_\nu\left(u\frac{r}{a}\right)\,\cos(\nu\varphi+\psi) & \text{for } 0 \le r \le a \\ A\,\frac{\mathrm{J}_\nu(u)}{\mathrm{K}_\nu(w)}\,\mathrm{K}_\nu\left(w\frac{r}{a}\right)\,\cos(\nu\varphi+\psi) & \text{for } r > a \end{cases},$$

where $\nu = 0, 1, 2, \ldots$ and $\psi \in \left\{0; \frac{\pi}{2}\right\}$.

Longitudinal electric field component:

$$\nabla\cdot\underline{\mathbf{D}}(\mathbf{r},t) = \nabla\cdot\left(\epsilon_0 n^2(\mathbf{r})\underline{\mathbf{E}}(\mathbf{r},t)\right) \approx \epsilon_0 n^2(\mathbf{r})\,\nabla\cdot\underline{\mathbf{E}}(\mathbf{r},t)$$
$$= \epsilon_0 n^2(\mathbf{r})\left(\frac{\partial\underline{\mathcal{E}}_x}{\partial x} + \frac{\partial\underline{\mathcal{E}}_y}{\partial y} - \mathrm{j}\,\beta\underline{\mathcal{E}}_z\right)\mathrm{e}^{\mathrm{j}(\omega t-\beta z)},$$

For $\underline{\mathcal{E}}_y = 0$, we find:

$$\underline{\mathcal{E}}_z = -\frac{\mathrm{j}}{\beta}\frac{\partial\underline{\mathcal{E}}_x}{\partial x} = -\frac{\mathrm{j}}{\beta}\left(\cos\varphi\frac{\partial\underline{\Psi}}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial\underline{\Psi}}{\partial\varphi}\right)$$

$\underline{\Psi}(r,\varphi)$ is continuous across $r = a$, and $\underline{\Psi}(r,\varphi)$ has the same φ-dependencies inside and outside the core, i.e., $\partial\underline{\Psi}/\partial\varphi$ is also continuous. $\partial\underline{\Psi}/\partial r$ must hence be continuous as well at $r = a$.

Linearly polarized (LP) modes

Characteristic equation of $LP_{\nu\mu}$ – modes:

Continuity of $\partial \underline{\Psi} / \partial \varphi$ at $r = a$ leads to

$$\frac{u\,J_\nu'(u)}{J_\nu(u)} = \frac{w\,K_\nu'(w)}{K_\nu(w)}$$

Using the recurrence relations of the Bessel functions, this can be written as

$$\frac{u\,J_{\nu-1}(u)}{J_\nu(u)} = -\frac{w\,K_{\nu-1}(w)}{K_\nu(w)} \text{ for } \nu \geq 0$$

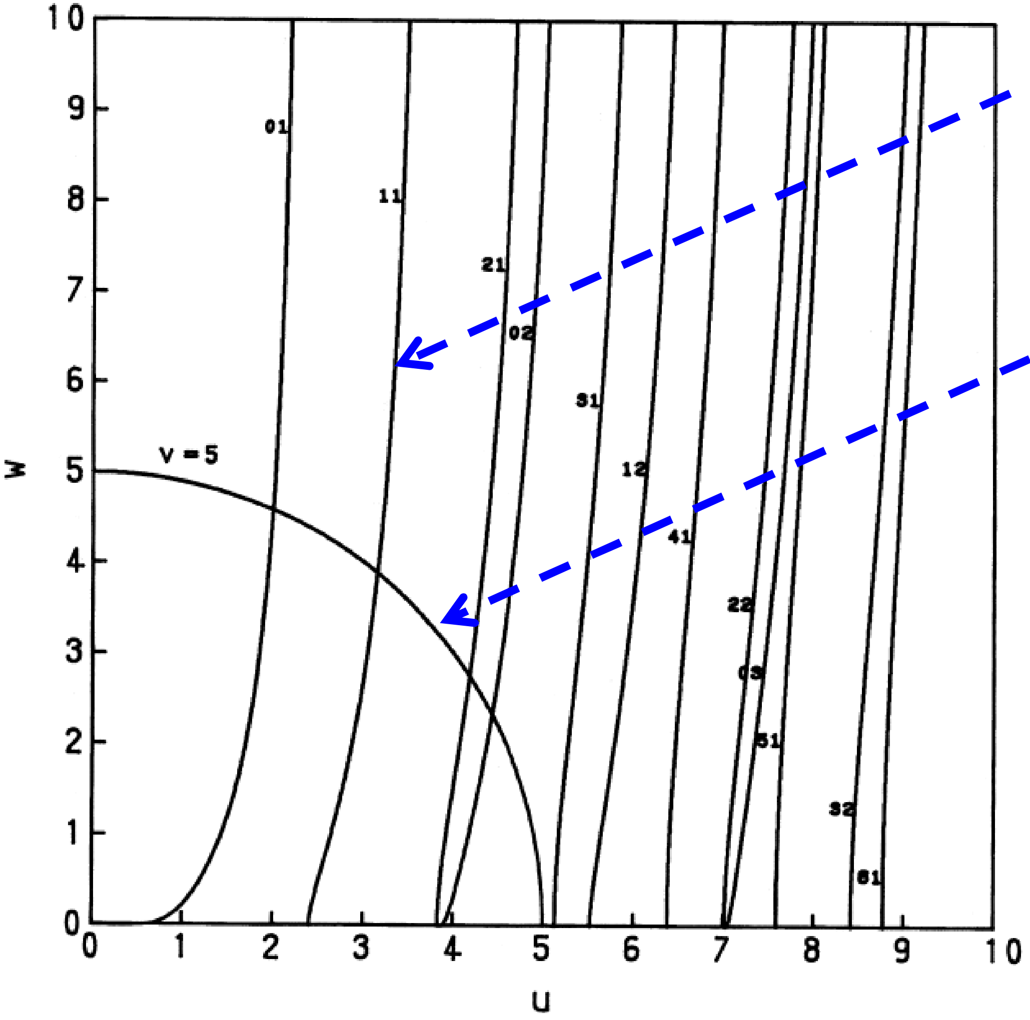
$$\frac{u\,J_1(u)}{J_0(u)} = \frac{w\,K_1(w)}{K_0(w)} \text{ for } \nu = 0$$

where

$$u^2 + w^2 = V^2$$

Recurrence relations of Bessel functions...

LP-modes: Solution of characteristic equation



Okamoto, Fundamentals of Optical Waveguides

Numerical solution of

$$\frac{u\,\mathrm{J}_\nu'(u)}{\mathrm{J}_\nu(u)} = \frac{w\,\mathrm{K}_\nu'(w)}{\mathrm{K}_\nu(w)}$$

Circular arc

$$u^2 + w^2 = V^2$$

Mode designation:

$$\mathrm{LP}_{\nu\mu}$$

ν: Azimuthal dependence: cos(νφ) or sin(νφ); 2ν = number of intensity maxima along the circumference

μ: Denotes the various solutions for a given ν and can be identified with the number of intensity maxima in radial direction

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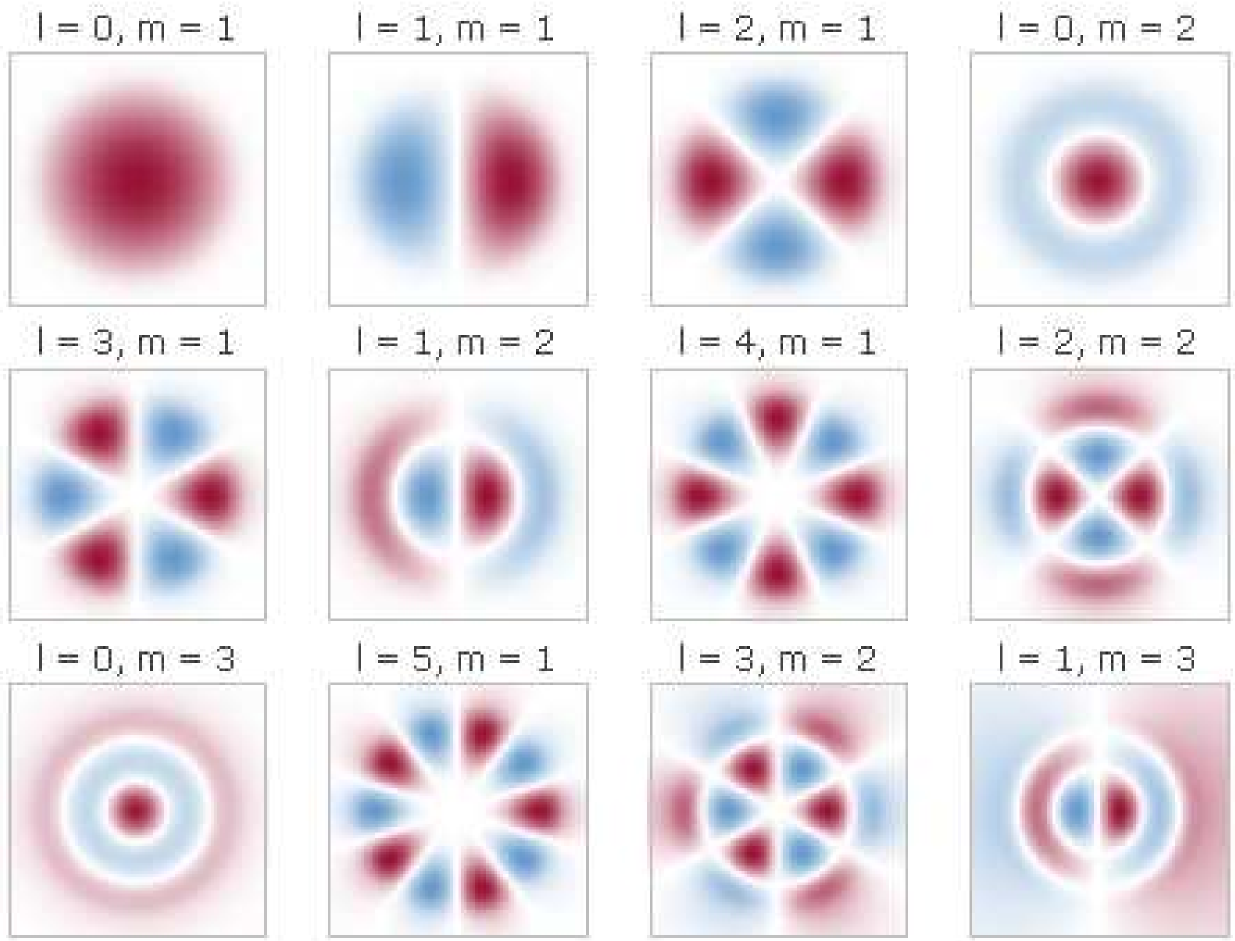
LP-modes: Mode fields



http://www.rp-photonics.com/multimode_fibers.html

- Different notation: $\nu = l$; $\mu = m$

- Degeneracy of modes:

  $\nu = 0$: Twofold degenerate (two orthogonal polarizations)

  $\nu > 0$: Four-fold degenerate (cos($\nu\varphi$)- and sin($\nu\varphi$)-dependence, each in two orthogonal polarizations)

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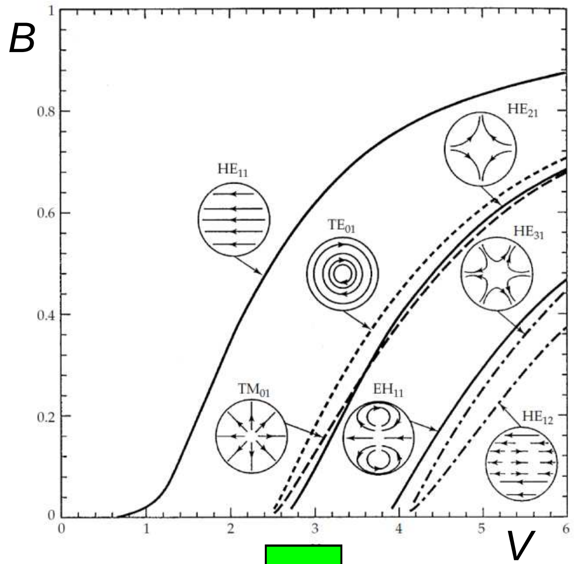
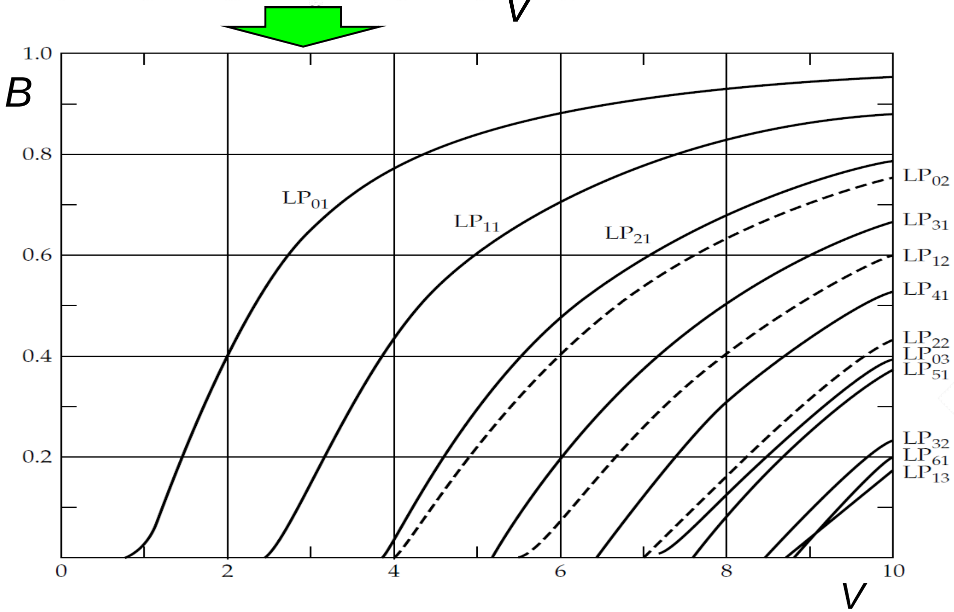
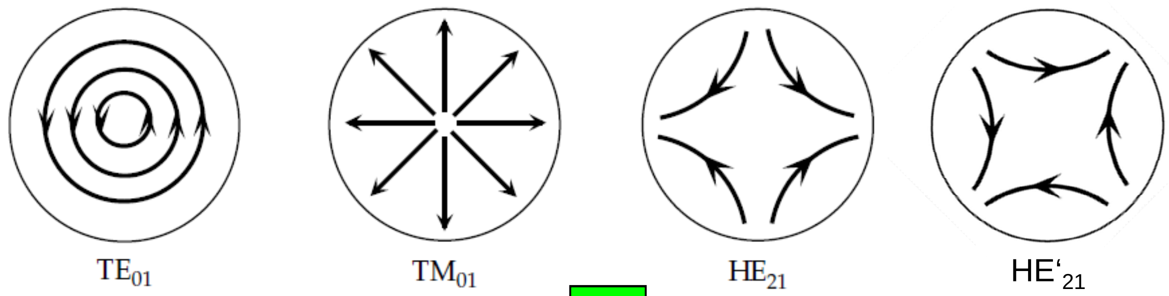
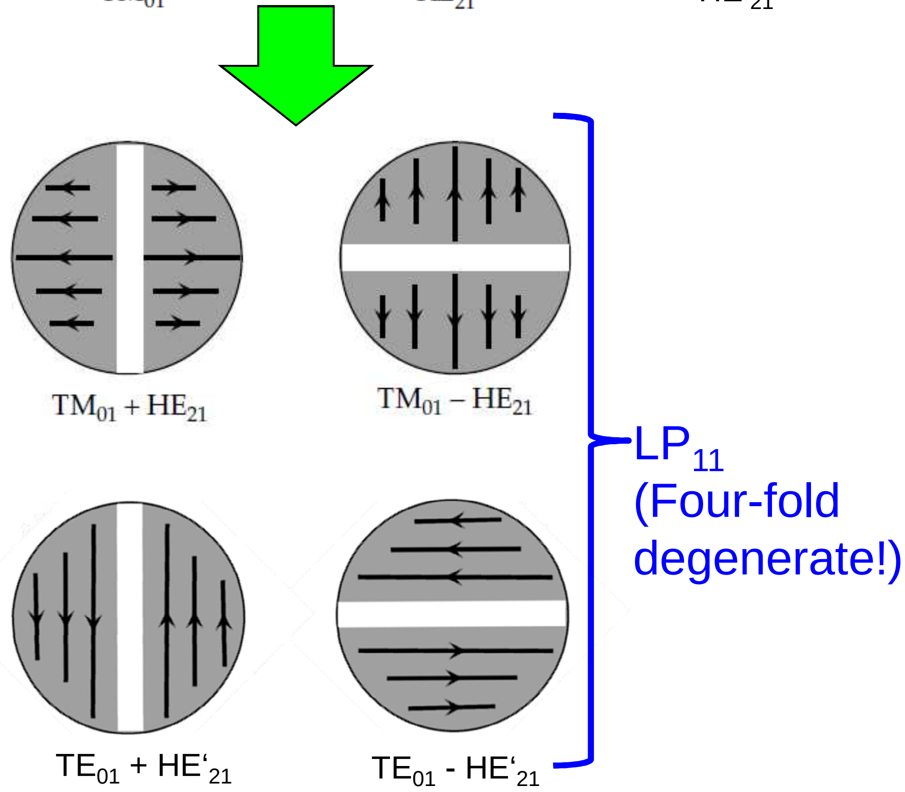
Linearly polarized (LP) and hybrid (TE, TM, EH, HE) modes









Chen, Foundations for Guided-Wave Optics, Wiley

LP-modes: Cut-off frequencies

Recall: Characteristic equation of LP – modes for $\nu > 0$:

$$\frac{u\,\mathrm{J}_{\nu-1}(u)}{\mathrm{J}_{\nu}(u)} = -\frac{w\,\mathrm{K}_{\nu-1}(w)}{\mathrm{K}_{\nu}(w)}$$

Bessel function plots

Waveguiding limit: Fields extend into the whole cladding region

$$w = a\sqrt{\beta^2 - n_2^2 k_0^2} \to 0$$
$$u = \sqrt{V^2 - w^2} \to V$$
$$w\frac{\mathrm{K}_{\nu-1}(w)}{\mathrm{K}_{\nu}(w)} \to 0,$$

$\Rightarrow$ $J_{\nu\text{-}1}(V) = 0$ at cut-off, i.e., the normalized cut-off frequency $V_{\nu\mu,\mathrm{C}}$ of the $LP_{\nu\mu}$-mode is given by the μ-th positive zero $j_{\nu\text{-}1,\mu}$ of $J_{\nu\text{-}1}$

$$V_{\nu\mu,\mathrm{C}} = j_{\nu-1,\mu}$$

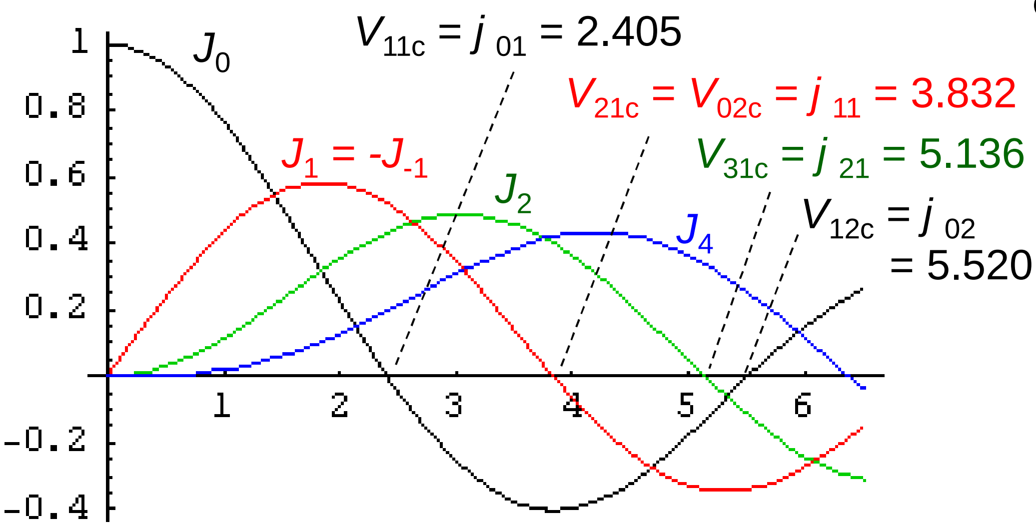
Note: $J_{-1}(u) = J_1(u)$, where u = 0 is counted as the first zero of $J_{\nu\text{-}1}(u)$ only for $\nu = 0$!

=> $LP_{0\mu}$ and $LP_{2,\mu\text{-}1}$-modes have the same cut-off frequencies!

HE vs. LP mode designation and cut-off frequencies

Normalized Frequency V	Traditional Mode Designation	LP Mode Designation	Additional Number of Modes	Total Number of Modes
0–2.4048	HE_{11}	LP_{01}	2	2
2.4048–3.8317	TE_{01}, TM_{01}, HE_{21}	LP_{11}	4	6
3.8317–5.1356	EH_{11}, HE_{31}, HE_{12}	LP_{21}	4	10
		LP_{02}	2	12
5.1356–5.5201	EH_{21}, HE_{41}	LP_{31}	4	16
5.5201–6.3802	TE_{02}, TM_{02}, HE_{22}	LP_{12}	4	20

Chen, Foundations for Guided-Wave Optics, Wiley



General mode equivalences for weakly guiding fibers:

$$LP_{0\mu} \Leftrightarrow HE_{1\mu}$$

$$LP_{1\mu} \Leftrightarrow HE_{2\mu},\ TE_{0\mu},\ TM_{0\mu}$$

$$LP_{\nu\mu} \Leftrightarrow HE_{\nu+1,\mu},\ EH_{\nu-1,\mu} \quad \text{for } \nu > 1$$

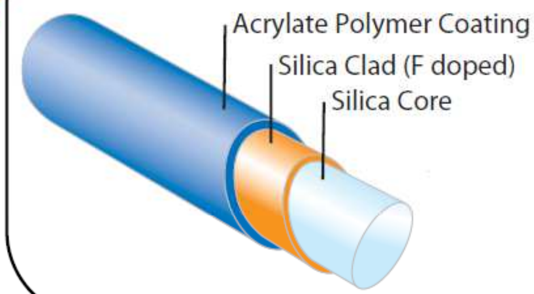
Number of guided modes

Product Specification Sheet

THORLABS

Step Index Multimode Fiber

AFS50/125Y



Description

This low loss, step index, multimode fiber is excellent for holmium and erbium laser delivery. The low hydroxyl ion content providing high transmission efficiency and has a useful spectral transmission range from 400 to 2400 nm.

Specifications

Specification	Value
Core Diameter	50 µm ± 2%
Clad Diameter	125 +1/-3 µm
Acrylate Diameter	250 µm ± 5%
Temperature Operating Range	-40 to 85 °C
Minimum Bend Radius	
Recommended Short Term	12 mm
Recommended Long Term	24 mm
Proof Test (Bend Method)	50 kpsi

Specification	Value
Buffer/Jacket	Acrylate
Numerical Aperture	0.22 ± 0.02
Full Acceptance Angle	25°
Core/Clad Ratio:	2.5
Maximum Attenuation:	9 dB/km @650 nm 4 dB/km @ 1500 nm

=> How many guided LP modes exist?

Number of guided modes in step-index fibers

Approximation for large zeros of Bessel functions:

When ν is fixed, $s >> \nu$ and $\mu = 4\nu^2$

9.5.12

$$j_{\nu,s}, y_{\nu,s} \sim \beta - \frac{\mu-1}{8\beta} - \frac{4(\mu-1)(7\mu-31)}{3(8\beta)^3} - \frac{32(\mu-1)(83\mu^2-982\mu+3779)}{15(8\beta)^5} - \frac{64(\mu-1)(6949\mu^3-1\ 53855\mu^2+15\ 85743\mu-62\ 77237)}{105(8\beta)^7} - \ldots$$

where $\beta=(s+\frac{1}{2}\nu-\frac{1}{4})\pi$ for $j_{\nu,s}$, $\beta=(s+\frac{1}{2}\nu-\frac{3}{4})\pi$ for $y_{\nu,s}$. With $\beta=(t+\frac{1}{2}\nu-\frac{1}{4})\pi$, the right of **9.5.12** is the asymptotic expansion of $\rho_\nu(t)$ for large t.

Notation: $j_{\nu,s}$ = s-th positive zero of $J_\nu(u)$

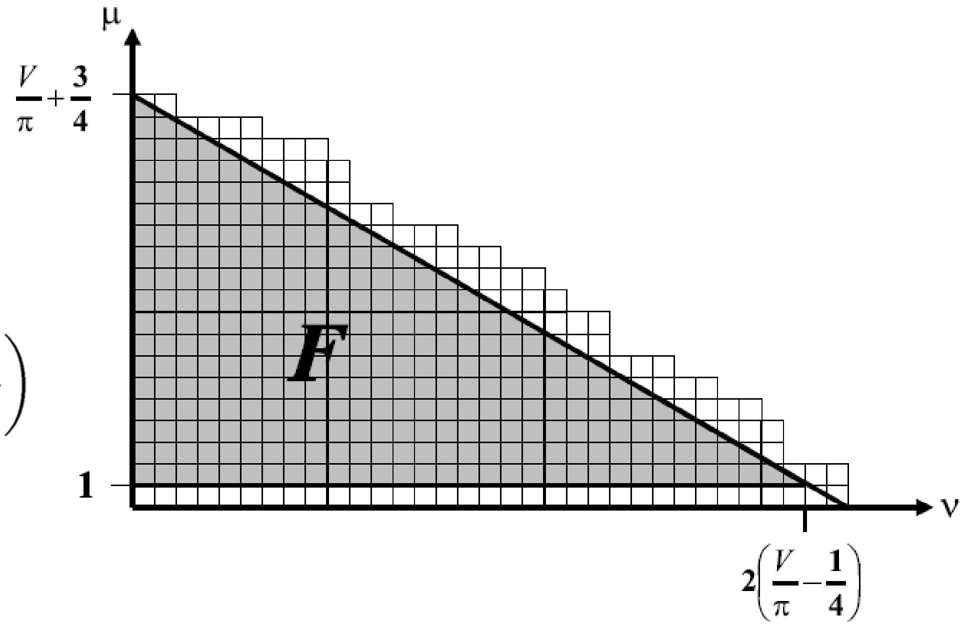
Abramowitz / Stegun, Handbook of Mathematical Functions

In our case:

$$j_{\nu-1,\mu} \approx \left(\mu + \frac{1}{2}\nu - \frac{3}{4}\right)\pi < V$$

Number of guided modes:

$$M_g \approx 4 \cdot \frac{1}{2} \cdot 2\left(\frac{V}{\pi} - \frac{1}{4}\right) \cdot \left(\frac{V}{\pi} - \frac{1}{4}\right) \approx \frac{4}{\pi^2}V^2 \approx \frac{V^2}{2}$$

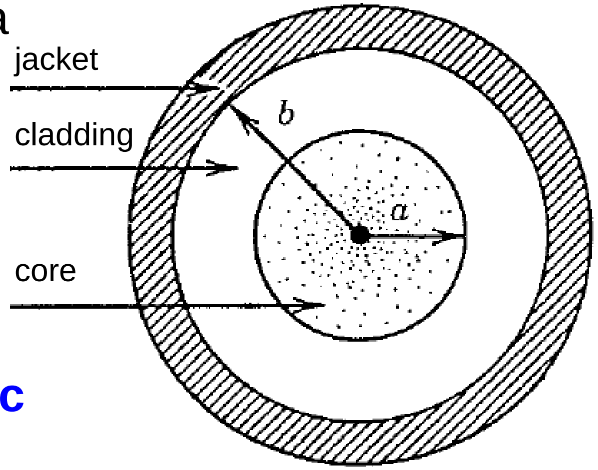


Graded-index fibers

Note: Many fibers of practical interest do not have a step-index profile!

Examples:

- Parabolic-profile step-index fibers ("GRIN-lenses")
- Manufactured fibers with non-perfect step-index profiles



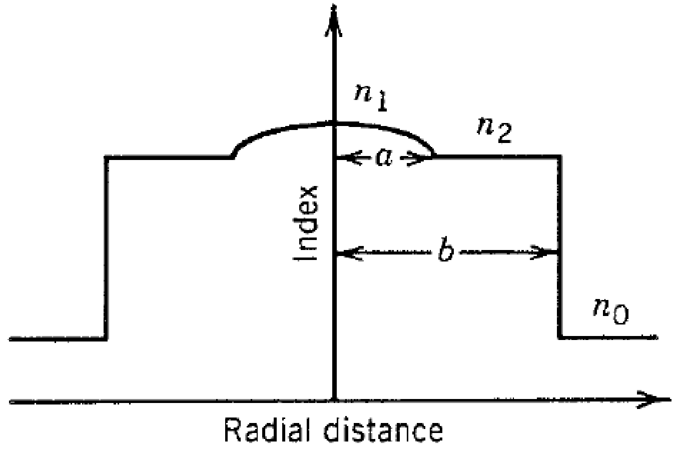
$\Rightarrow$ Power-law description of rotationally symmetric index profile:

$$n^2(r) = \begin{cases} n_1^2\left[1 - 2\Delta\, g\left(\frac{r}{a}\right)\right], & 0 \le r < a \\ n_1^2\left[1 - 2\Delta\right] = n_2^2, & a \le r < \infty \end{cases}$$

$$\Delta = \frac{n_1^2 - n_2^2}{2n_1^2}$$ **Relative index difference**

$$g\left(\frac{r}{a}\right) = \left(\frac{r}{a}\right)^q$$ **Profile function (power law)**

$$0 \le q < \infty.$$ **Profile parameter**



Note that closed-from solutions exist only for the special cases of step-index profiles (q $\rightarrow \infty$) and infinitely extended parabolic index profiles (q = 2)!

Infinitely extended parabolic index profile

Assumption: Infinitely extended parabolic index profile (unphysical!)

$$n^2(r) = n_1^2\,[1 - 2\Delta g(r/a)] \qquad g\left(\frac{r}{a}\right) = \left(\frac{r}{a}\right)^2, \text{ in } 0 \le r < \infty$$

Harmonic azimuthal dependence:

$$\underline{\Psi}\,(r,\varphi) = Q(r)\,\cos\,(\nu\varphi + \psi) \qquad \nu = 0, 1, 2, \ldots;\ \psi \in \left\{0; \frac{\pi}{2}\right\},$$

Solution for radial dependence (without derivation): Gauss-Laguerre modes

$$Q_{\nu\mu}(r) = \sqrt{\frac{2/n_1}{w_0^2\pi}}\sqrt{\frac{(\mu-1)!}{(\nu+\mu-1)!}}\left(\frac{2r^2}{w_0^2}\right)^{\nu/2} \mathrm{e}^{-r^2/w_0^2}\,\mathrm{L}_{\mu-1}^{(\nu)}\left(\frac{2r^2}{w_0^2}\right),$$

$$\mu = 1, 2, 3, \ldots$$

$$w_0^2 = \frac{a^2}{V/2}$$ **Gaussian field radius**

$\mathrm{L}_{\mu}^{(0)}$ **Ordinary Laguerre polynomials of degree μ**

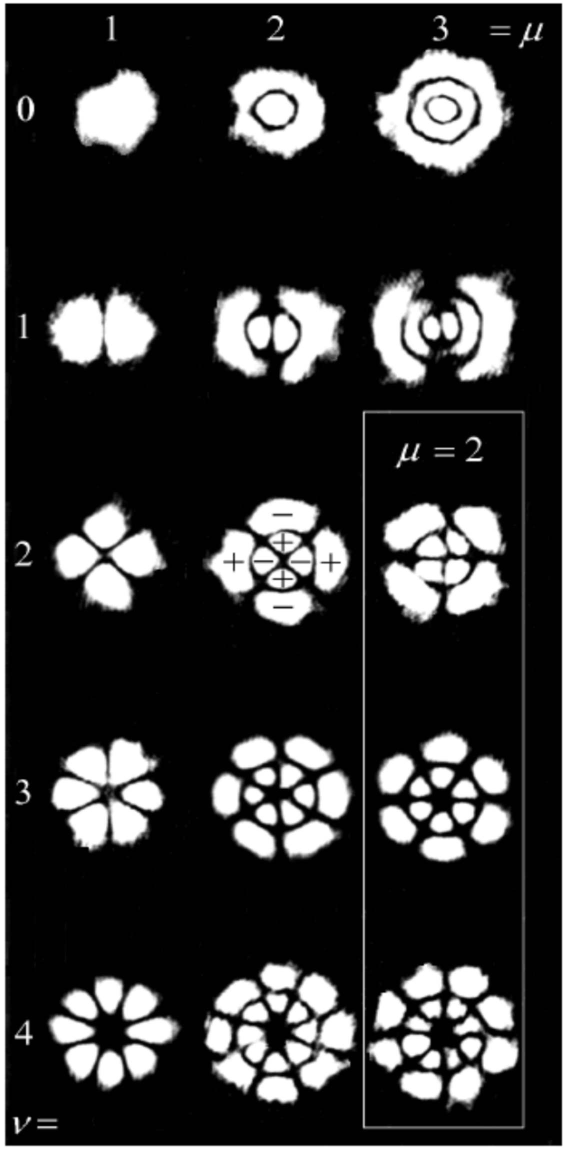
$\mathrm{L}_{\mu}^{(\nu)}$ **Modified Laguerre polynomials of degree μ and order ν**

Propagation constant:

$$\beta = n_1 k_0 \sqrt{1-2\delta}, \quad \frac{\delta}{\Delta} = \frac{m}{m_{\max}}, \quad m = \nu + 2\mu - 1, \quad m_{\max} = \frac{V}{2}; \quad w_0^2 = \frac{a^2}{V/2}.$$

i.e., all modes with identical m are degenerate!

Gauss-Laguerre mode fields



Mode designation (ν, μ):

ν: Azimuthal dependence: cos($\nu\varphi$) or sin($\nu\varphi$); 2ν = number of intensity maxima along the circumference

μ: Denotes the various solutions for a given ν and can be identified with the number of intensity maxima in radial direction (including the maximum at r = 0 for $\nu = 0$)

Degeneracy of modes:

$\nu = 0$: Twofold degenerate

(two orthogonal polarizations)

$\nu > 0$: Four-fold degenerate

(cos($\nu\varphi$)- and sin($\nu\varphi$)-dependence, each in two orthogonal polarizations)

Institute of Photonics and Quantum Electronics

Infinitely extended parabolic index profile: Number of guided modes

Cut-off: $\beta = k_2$

$$\Rightarrow m = m_{max}$$

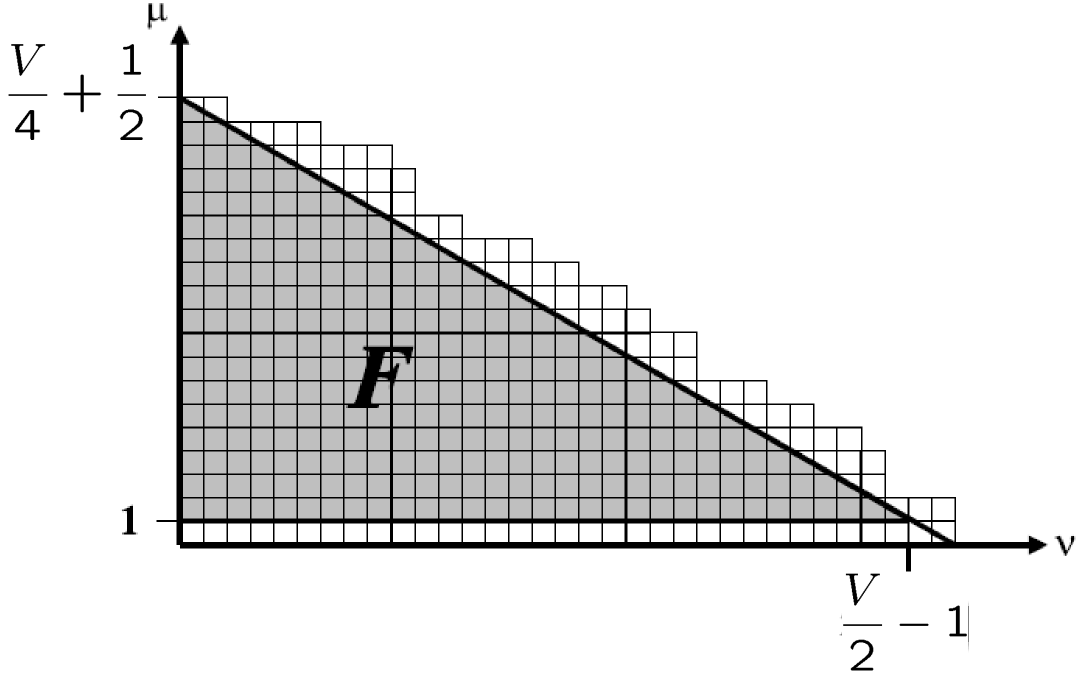
$$\Rightarrow \nu + 2\mu - 1 = \frac{V}{2}$$

Guided modes must fulfill:

$$\nu + 2\mu - 1 < \frac{V}{2}$$

$$\nu = 0, 1, 2, \ldots$$

$$\mu = 1, 2, 3, \ldots$$

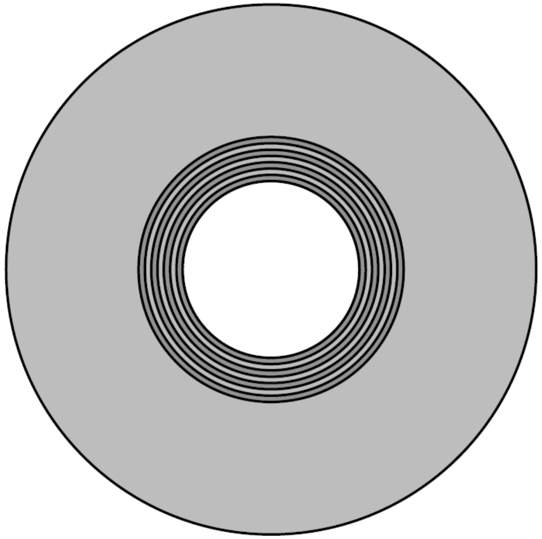


Number of guided modes:

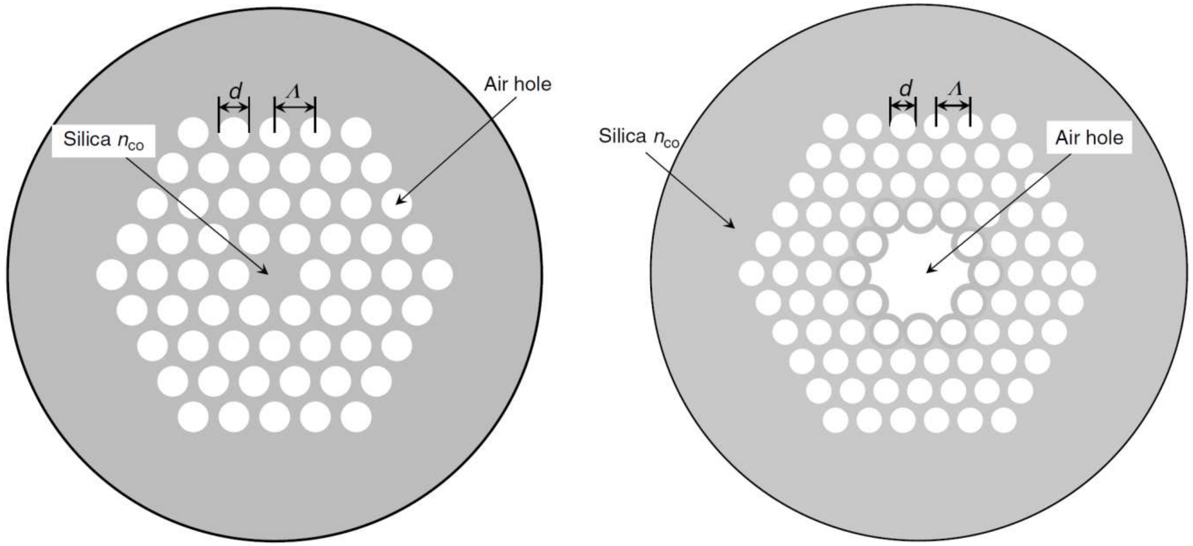
$$M_g \approx 4 \cdot \frac{1}{2} \cdot \left(\frac{V}{2} - 1\right) \cdot \left(\frac{V}{4} + \frac{1}{2}\right) \approx \frac{V^2}{4}$$

i.e., in comparison with a step-index fiber of same V, the parabolic index profile accepts approximately only half the number of guided modes!

Photonic-bandgap fibers



Bragg fiber



Photonic-crystal fibers

- Cladding: Concentric rings / periodic structures of high- and low-index materials

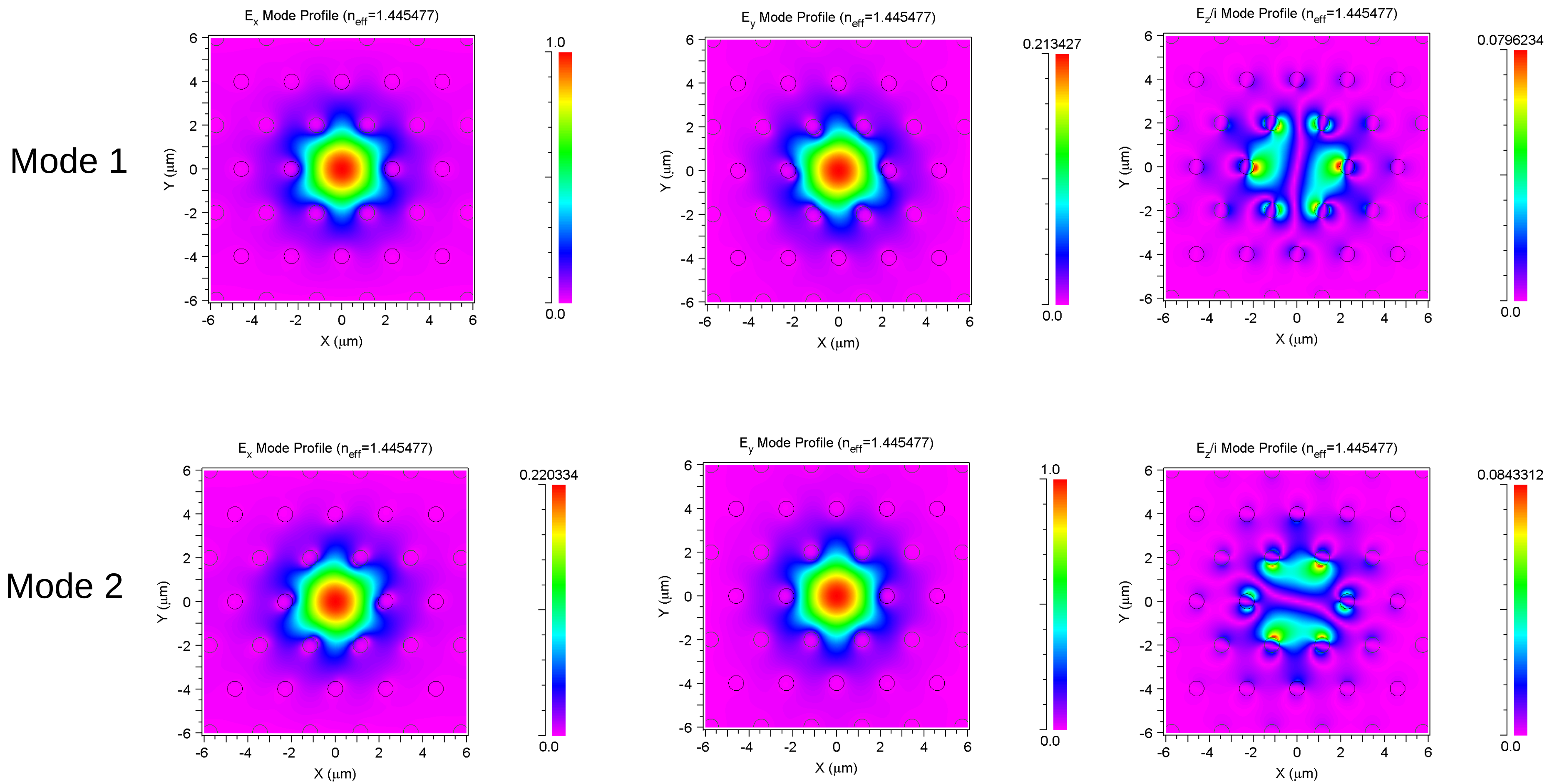
⇒ Guidance due to multiple reflections ("photonic bandgap")

- Core: Can be hollow!
- Hollow-core fiber: Little interaction of guided light with fiber material

⇒ Low absorption for wavelengths where no transparent fiber materials are available.

Weak nonlinear effects, high power levels (e.g. for material processing)

Photonic crystal fiber: Fundamental modes ?

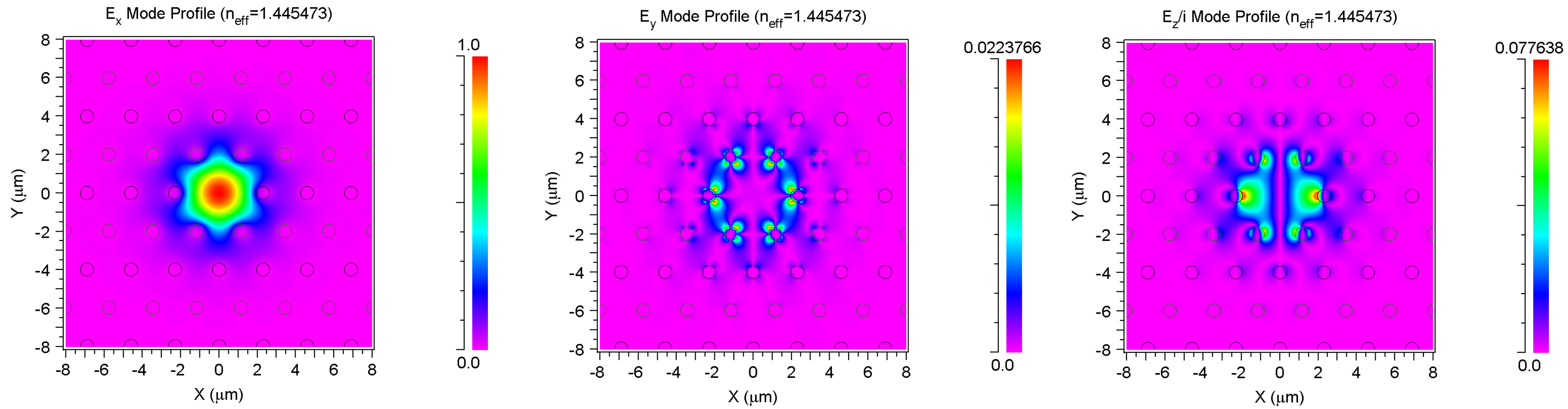


Looks „suspicious“: Mode field does not have the same symmetry properties as the structure!

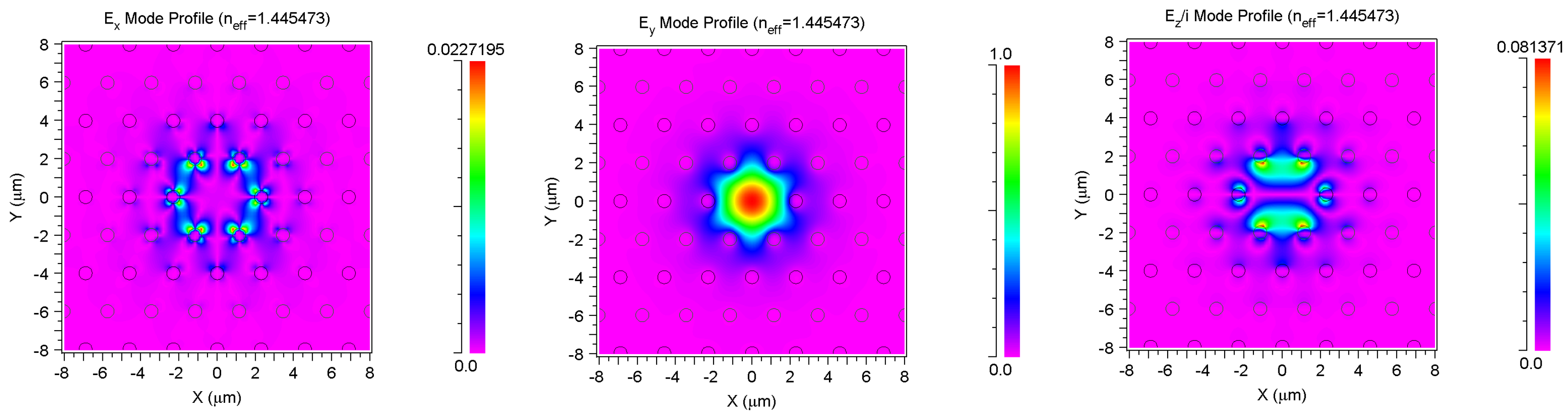
$\Rightarrow$ Choose finer discretization; check convergence!

Photonic crystal fiber: Fundamental modes !

Mode 1



Mode 2

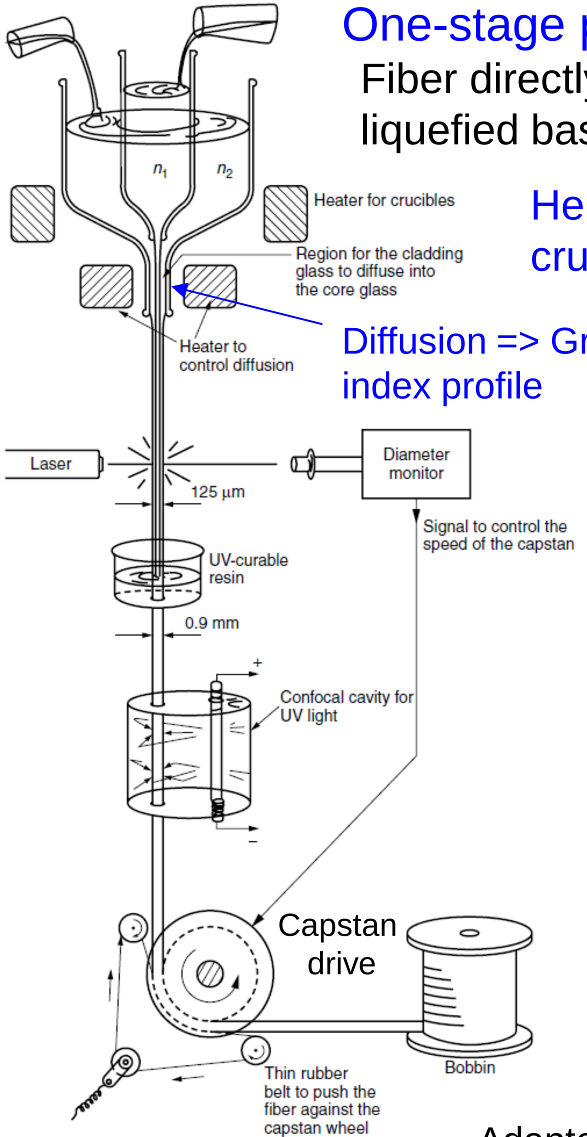


Drawing of optical fibers: One- and two-stage processes

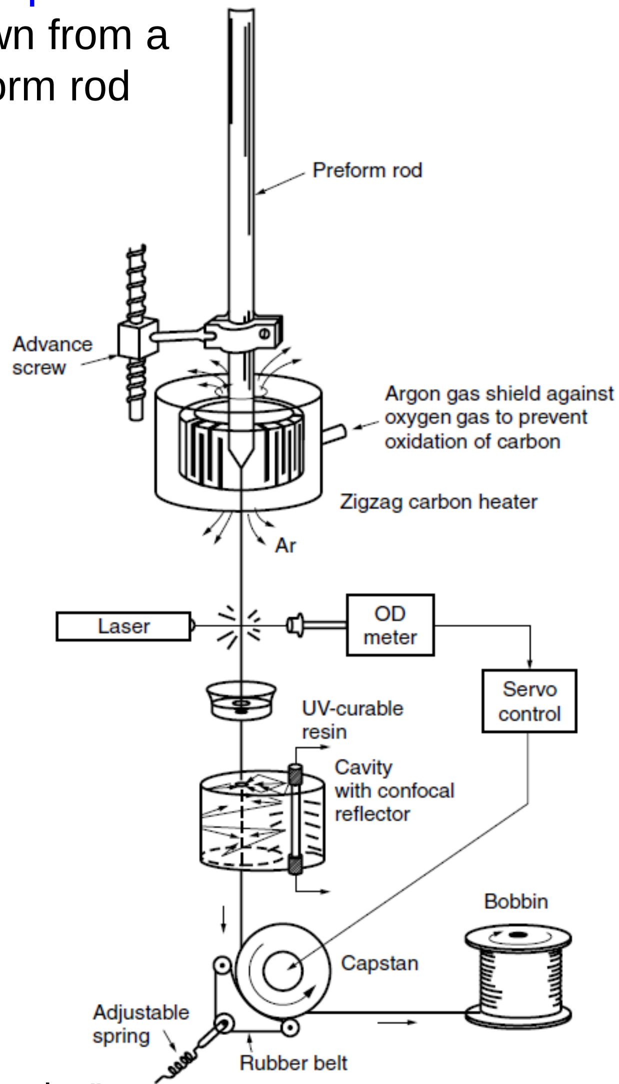
One-stage process:
Fiber directly drawn from liquefied base materials

Here: Double-crucible method

Diffusion => Graded index profile



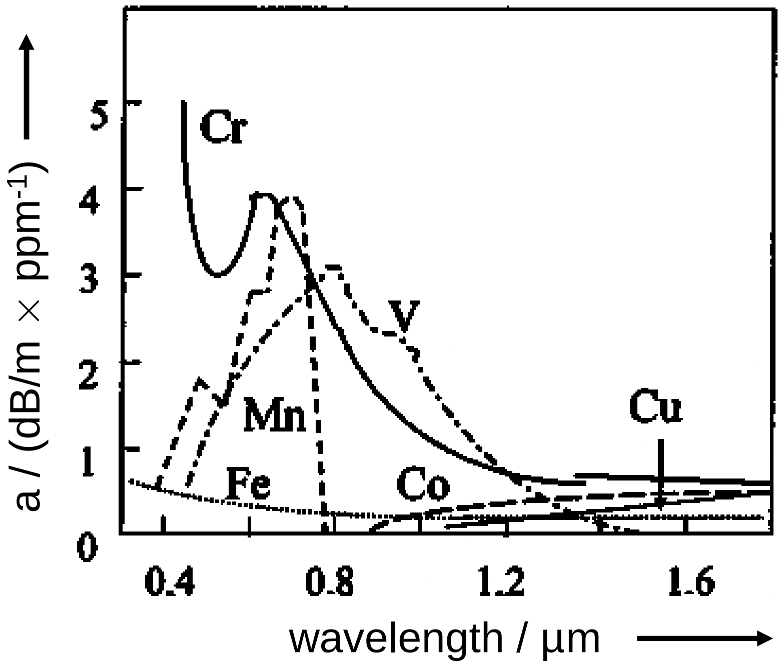
Two-stage process:
Fiber drawn from a solid preform rod

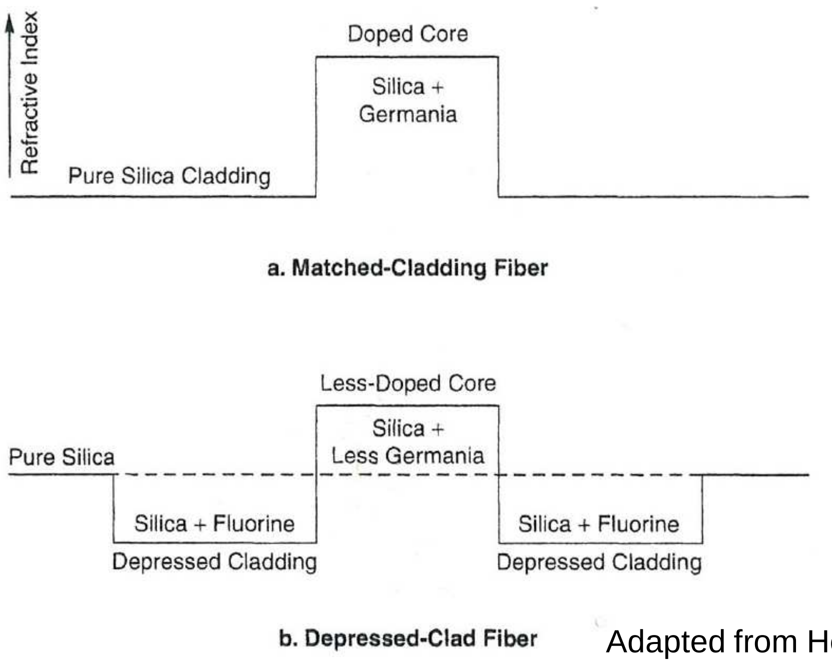


Adapted from Iizuka, "Elements of Photonics"

Low-loss silica glass fibers

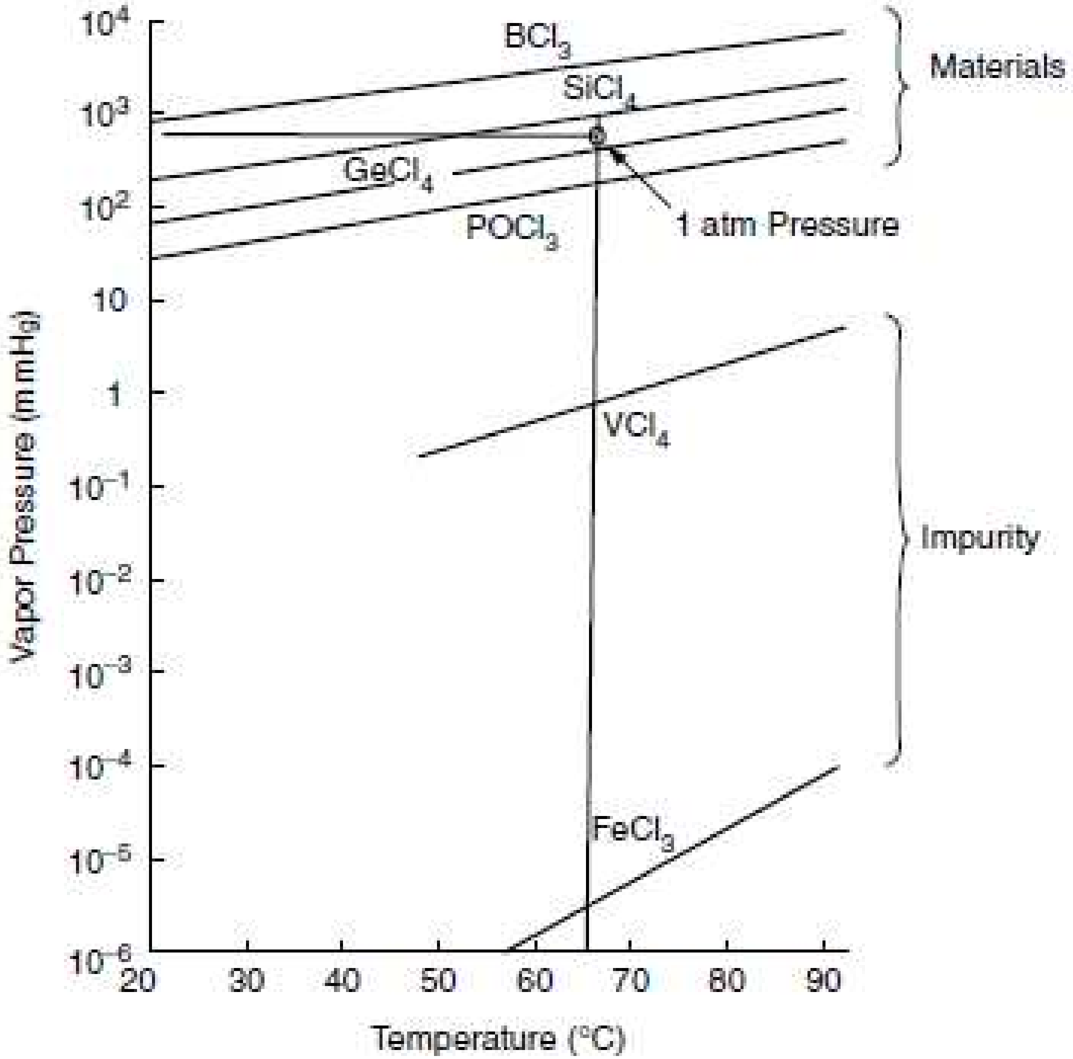
- Telecom fibers consist of highly pure fused silica (amorphous SiO_2, “Quarzglas”)
- Purity in the sub-ppb-range needed! => Material is made synthetically by chemical vapor deposition (CVD):
  - Oxidizing silicon tetrachloride ($SiCl_4$); this yields highly pure white SiO_2 soot, which is melted into the preform
  - Dopants allow to increase/decrease refractive index:
    - Oxidation of $GeCl_4$ ($POCl_3$) leads to formation of GeO_2 (P_2O_5) => Increase of refractive index
    - Flourine (F) doping allows to reduce the refractive index





Adapted from Hecht, “Understanding Fiber Optics”

Synthesis of highly pure silica by CVD



Iizuka, "Elements of Photonics"

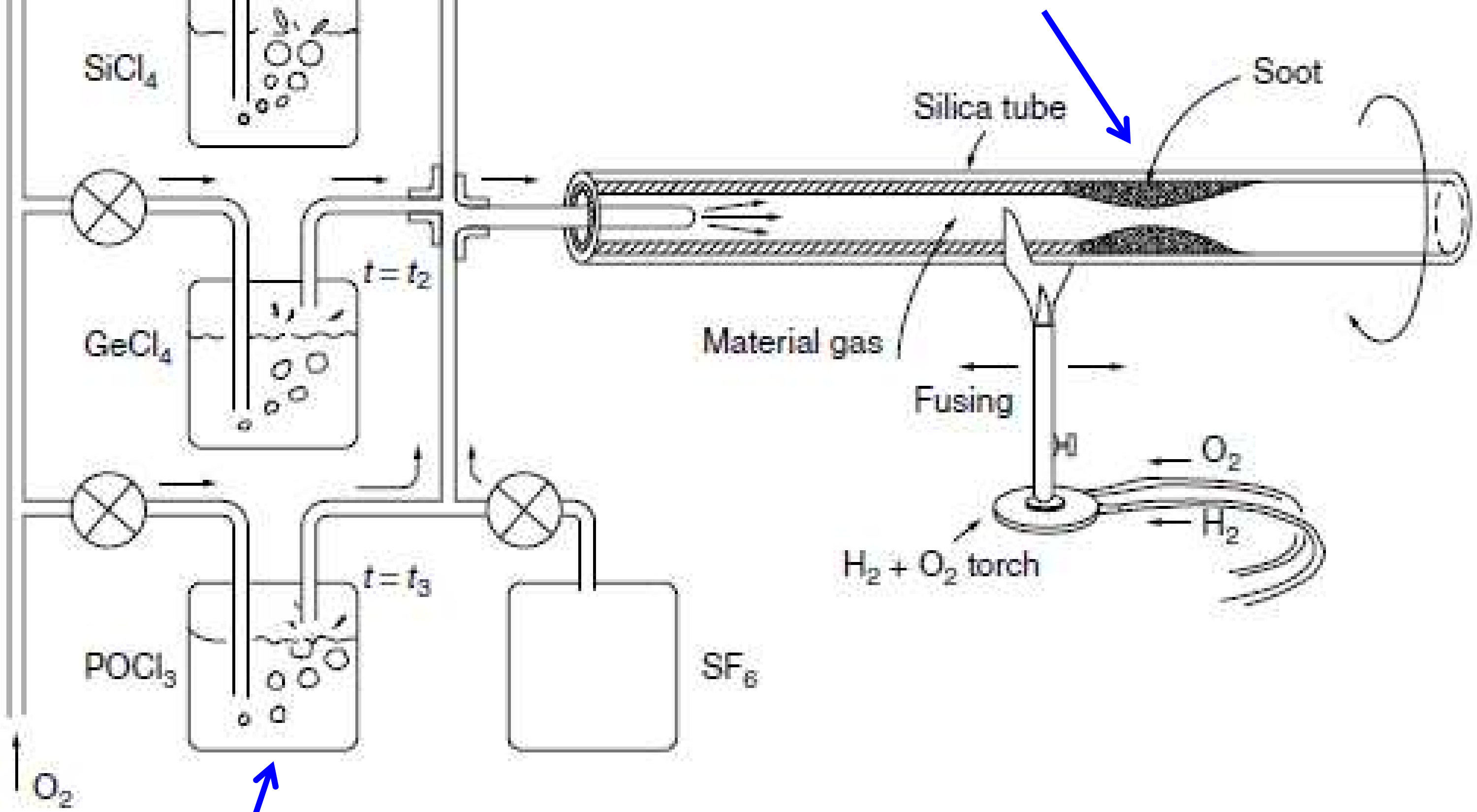
- Vastly different in vapor pressures of impurity-metal halides and $SiCl_4$, $GeCl_4$ or $POCl_3$

- Evaporation of, e.g., $SiCl_4$ at 65° and normal pressure leaves unwanted VCl_4, $FeCl_3$ etc. in the liquid!

$\Rightarrow$ Synthesis of highly pure SiO_2 possible

Silica preform fabrication: Inside vapor deposition (IVD)

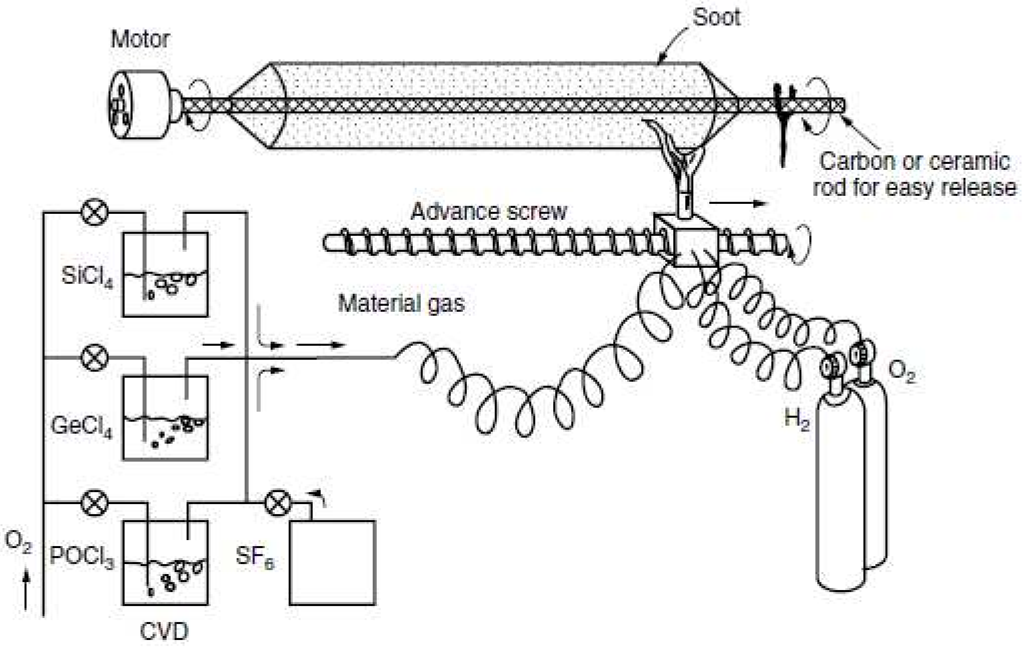
Silica soot deposited to the inner walls of a silica rod, fused to the tube wall as the flame passes!



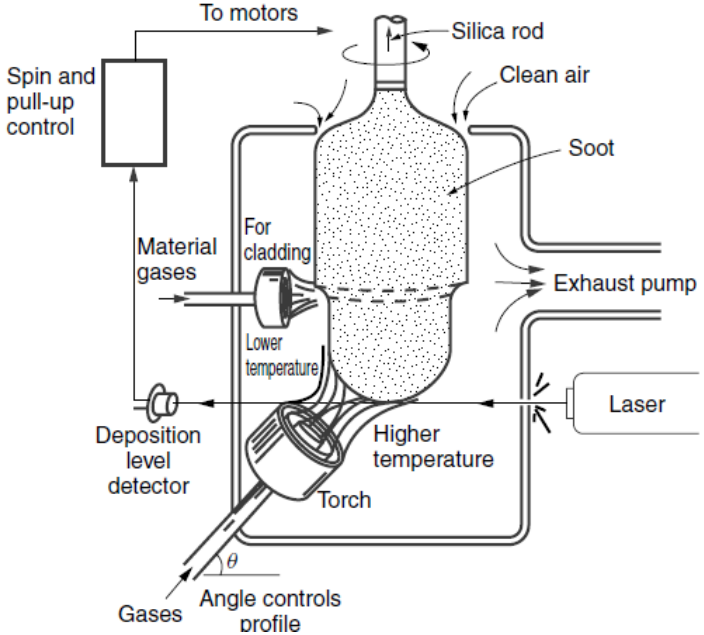
Material gases evaporated under well-controlled conditions (temperature and pressure)

Iizuka, "Elements of Photonics"

Outside vapor deposition (OVD)



Vapor Axial Deposition (VAD)



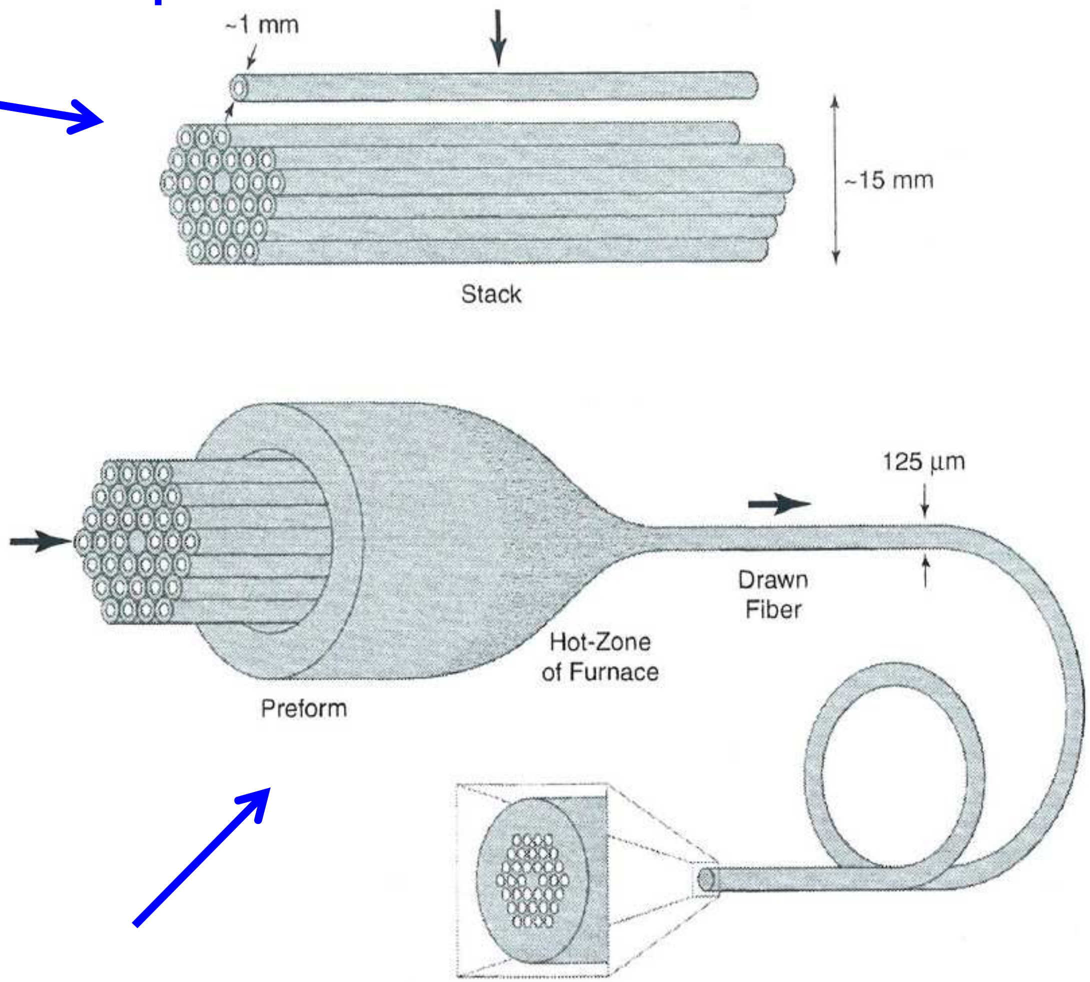
- Material gases directly injected into the hydrogen flame
- Deposited soot contains OH^--ions which lead to optical loss and must hence be removed by flushing the preform with Cl_2 at elevated temperatures before melting the soot into a solid preform
- VAD allows for very long preforms contrast to OVD and IVD

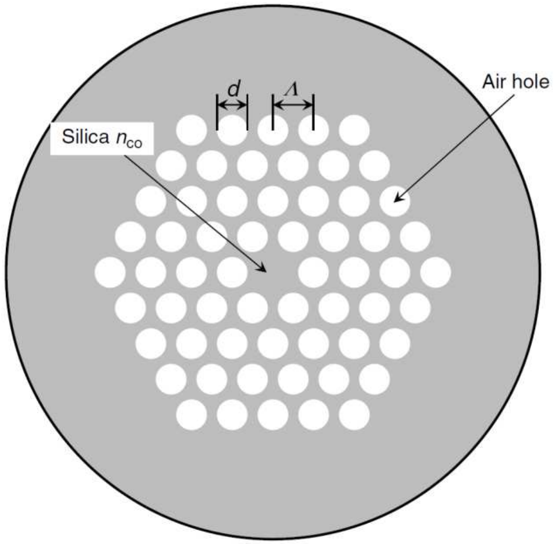
Iizuka, “Elements of Photonics”

Preform fabrication for microstructured fibers

Fabrication by drawing from solid preform:

Preform: Stack of hollow glass tubes and rods, enclosed by outer tube



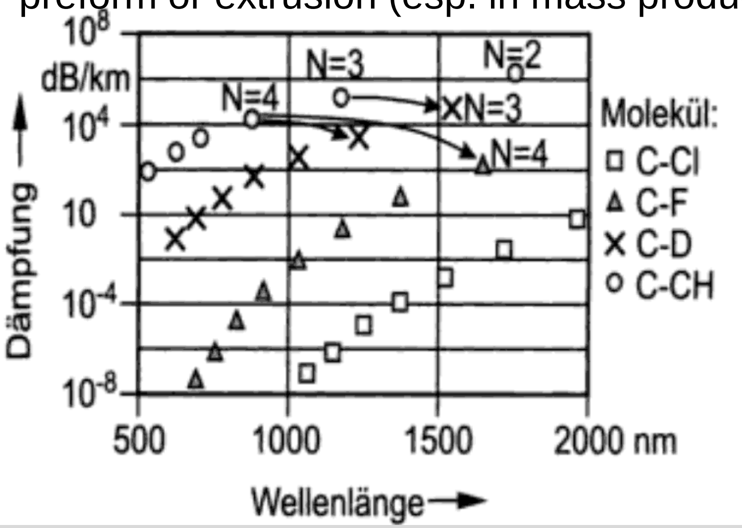


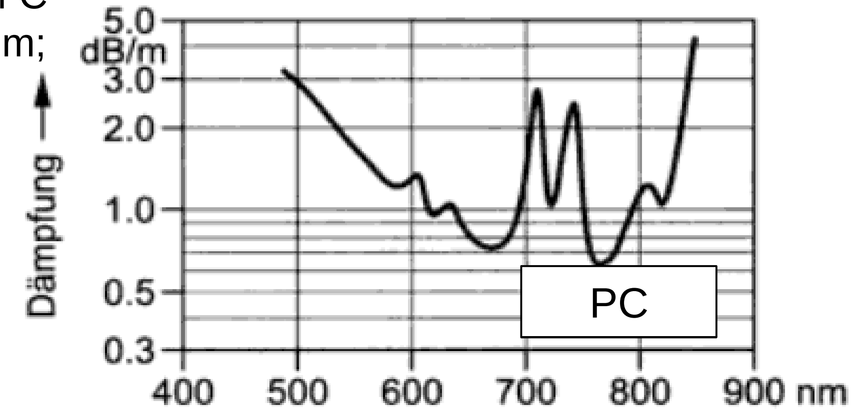
Melting/drawing yields microstructured ("holey") fiber

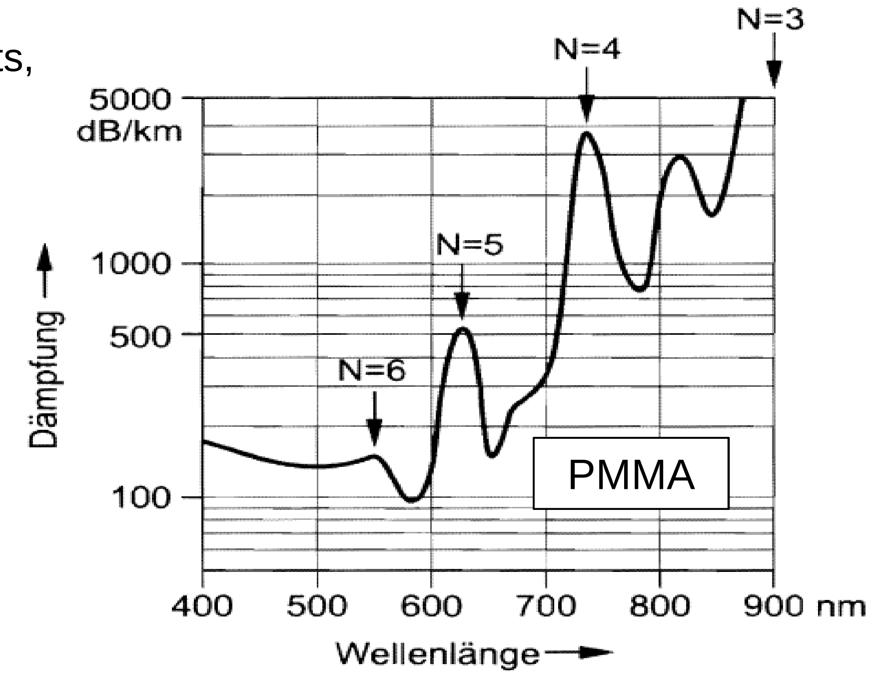
Hecht, "Understanding Fiber Optics"

Polymer fibers

- Large infrared material absorption due to overtones of C-H-bond oscillations (fundamental oscillation at 3390 nm; overtones at ~1700 nm, 850 nm …)
- Absorption can be reduced by using fluorinated polymers (lower oscillation frequency!)
- Losses today: > 20 dB/km (850 nm / 1300 nm)
- **Advantages:** Large diameters (85 um … > 3mm) => Relaxed mechanical tolerances
- **Applications:**
  - Image transmission bundles
  - Short data links in automotive, optical interconnects, home installations
- **Fabrication of polymer fibers:** Drawing from solid preform or extrusion (esp. in mass production)





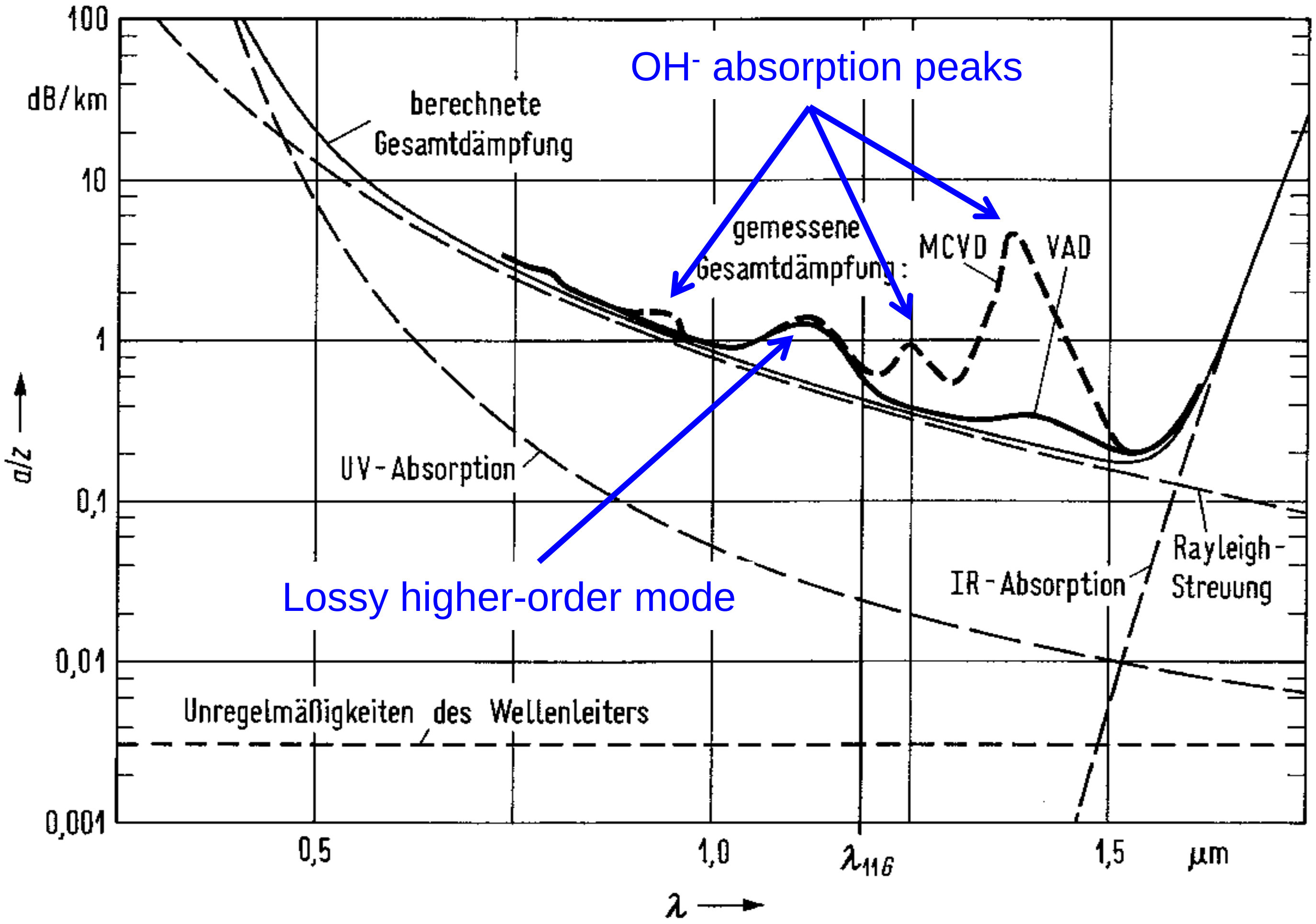


Voges/ Petermann: Optische Kommunikationstechnik

Signal propagation in fused-silica fibers: Loss spectrum

Note: $P(z) = P_0\,\mathrm{e}^{-\alpha z}, \qquad \frac{a}{z} = 10\lg\frac{P_0}{P(z)} = \alpha\,10\lg\mathrm{e} = 4.34\,\alpha$

Attenuation of Ge-doped (Δ=0.25 %) singlemode fibre:



VAD = Vapor axial deposition

MCVD = Modified chemical vapor deposition

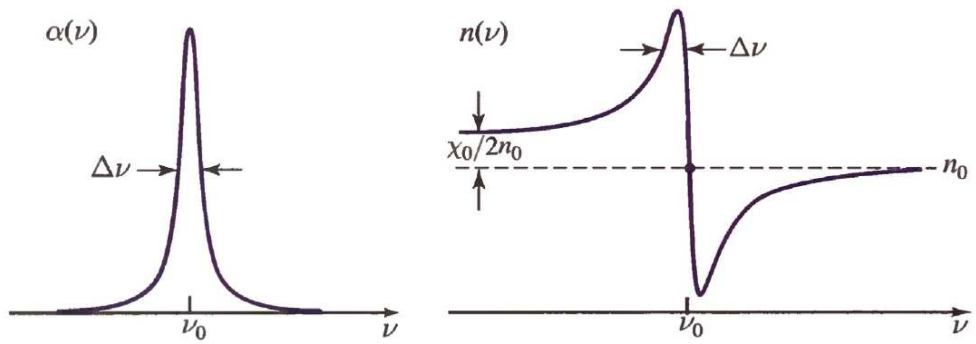
Basic loss mechanisms:

- Intrinsic absorption
- Extrinsic absorption
- Scattering

Sources of loss in silica fibers I

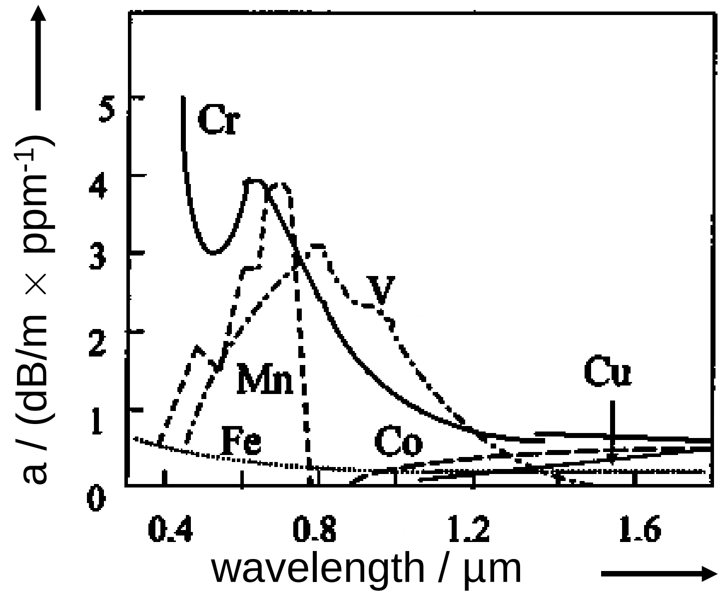
Intrinsic absorption:

- Unavoidable absorption by fused silica (SiO_2)
- Electronic resonances in the ultraviolet (λ < 0.4 µm) => UV absorption
- Vibrational resonances in the infrared (λ > 7 µm) => IR absorption
- Intrinsic loss below 0.1 dB/km in the wavelength range between 1.3 µm and 1.6 µm



Extrinsic absorption due to impurities

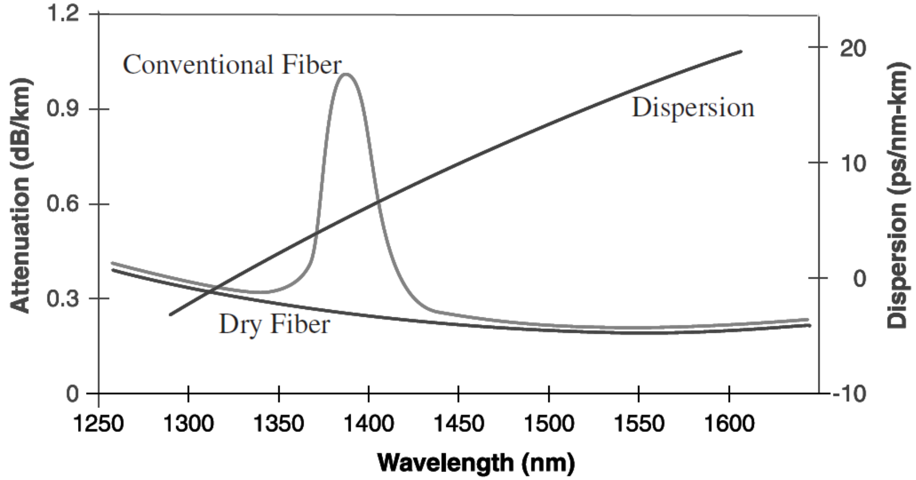
- Transition **metals** Fe, Cu, Co, Ni, Mn and Cr absorb strongly in the wavelength range between 1.3 µm and 1.6 µm => Need to reduce concentration below 1 ppb
- **OH-absorption** due to water / OH^- ions: Vibrational resonance of the OH-bond occurs near 2.73 µm; harmonic and combination tones with silica resonances produce absorption around 1.39 µm, 1.24 µm and 0.95 µm
- Standard fibers: OH^- concentration below 10^{-8}; 1.39 µm absorption peak below 1 dB / km



Sources of loss in silica fibers II

- **“Dry fiber”:** OH^- concentration reduced to such low levels that the absorption peak almost disappears; can transmit signals over the entire band between 1.3 µm and 1.6 µm and is marketed under the trade name

AllWave® Fiber
Zero Water Peak
The New Standard for Single-Mode Fiber



Rayleigh scattering:

- Amorphous material => Random fluctuations of the refractive index on a scale much smaller than the wavelength
  => Rayleigh scattering
- Loss ~ λ^{-4}, amounts to around 0.12 – 0.16 dB/km at 1550 nm
- Dominant loss mechanism in state-of-the art fibers!

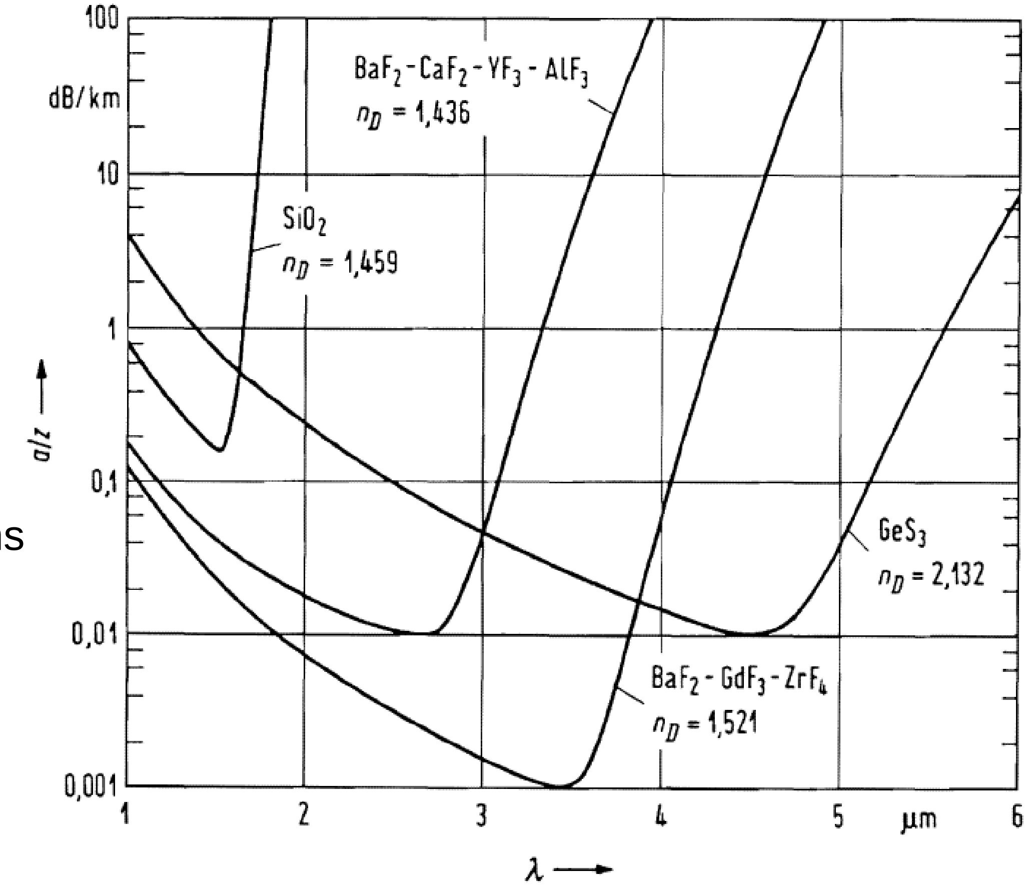
Waveguide imperfections:

- Imperfections at **core-cladding interface**, e.g., random core radius variations (typically below 1%); resulting fiber loss is typically below 0.03 dB/km
- **Microbends**: Random axial distortions; can occur when the fiber is “squeezed”, e.g. inside a cable assembly; can be minimized by appropriate cabling (jelly) and by keeping the normalized frequency V as large as possible (typically between 2.0 and 2.4)
- **Macrobends**: Bend radii of typically a few millimeters (“visible with the bare eye”); bending loss is usually negligible for single mode fibers and bend radii larger than 5 mm

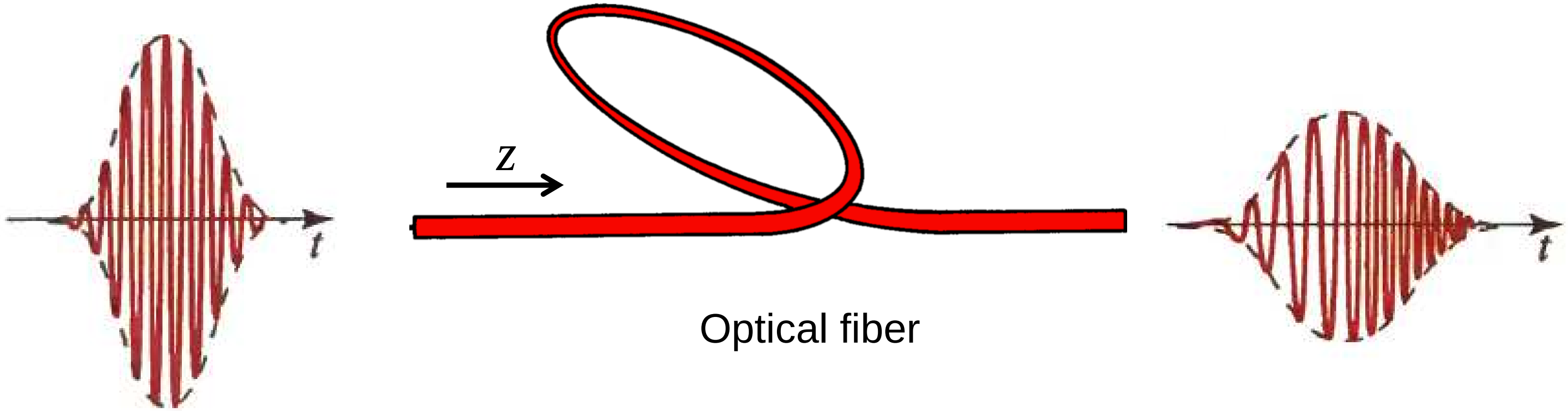
Alternatives to silica glass ?

- Minimize Rayleigh scattering by using bigger wavelengths => Needs low infrared absorption!
- Materials with low intrinsic IR losses exist, but extrinsic absorption lead to attenuation values much larger than those of silica fibers!
- In addition: Devices (especially fast detectors) not available at wavelengths beyond 2 µm!

⇒ **Currently no alternative to silica glass fibers!**



Signal propagation in dispersive single-mode fibers



$$\underline{a}\,(0,t) = \underline{A}\,(0,t)\exp\left(\mathrm{j}\,\omega_c t\right)$$

$$\underline{a}\,(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{\tilde{A}}\,(0,\omega-\omega_c)\,\mathrm{e}^{-\mathrm{j}\,\beta(\omega)z}\,\mathrm{e}^{\mathrm{j}\,\omega t}\,\mathrm{d}\omega$$

$$\underline{\tilde{a}}\,(0,\omega) = \underline{\tilde{A}}\,(0,\omega-\omega_c)$$

Propagation: $\mathrm{e}^{-\mathrm{j}\,\beta(\omega)z}$

$$\underline{\tilde{a}}\,(z,\omega) = \underline{\tilde{A}}\,(0,\omega-\omega_c)\,\mathrm{e}^{-\mathrm{j}\,\beta(\omega)z}$$

Taylor expansion of mode propagation constant for narrowband (slowly varying) signal:

$$\beta(\omega) = \frac{\omega}{c} n_e(\omega) \approx \beta_c^{(0)} + (\omega-\omega_c)\beta_c^{(1)} + \frac{(\omega-\omega_c)^2}{2!}\beta_c^{(2)} + \frac{(\omega-\omega_c)^3}{3!}\beta_c^{(3)} + \ldots$$

where $$\beta_c^{(i)} = \left.\frac{\mathrm{d}^i\beta(\omega)}{\mathrm{d}\omega^i}\right|_{\omega=\omega_c}$$

Quantitative analysis of chromatic dispersion

Electric field of an LP-mode propagationg along a fiber

$$\underline{E}_x(\mathbf{r},t) = \underbrace{\underline{\Psi}(x,y)}_{\text{lateral mode field}} \times \underbrace{\underline{A}(z,t)}_{\text{slowly varying envelope}} \times \underbrace{\exp\left(\mathrm{j}\left(\omega_c t - \beta_c z\right)\right)}_{\text{optical carrier}}$$

$$= \underline{\Psi}(x,y)\, \underline{a}(z,t)$$

z- and *t*-dependence and slowly varying envelope ansatz:

$$\underline{a}(z,t) = \underline{A}(z,t)\exp\left(\mathrm{j}\left(\omega_c t - \beta_c z\right)\right) \quad \circ\!\!-\!\!\bullet \quad \underline{\tilde{a}}(z,\omega) = \underline{\tilde{A}}(z,\omega-\omega_c)\exp\left(-\mathrm{j}\,\beta_c z\right)$$

Propagation:

$$\underline{\tilde{a}}(z,\omega) = \underline{\tilde{a}}(0,\omega)\exp\left(-\mathrm{j}\,\beta(\omega)\,z\right)$$

$$\beta(\omega) \approx \beta_c + (\omega-\omega_c)\beta_c^{(1)} + \frac{(\omega-\omega_c)^2}{2!}\beta_c^{(2)} + \frac{(\omega-\omega_c)^3}{3!}\beta_c^{(3)} + \ldots\,,$$

Evolution of slowly varying envelope in the frequency domain:

$$\underline{A}(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underline{\tilde{A}}(0,\omega)\exp\left(-\mathrm{j}\left(\beta_c^{(1)}\omega z + \frac{\beta_c^{(2)}}{2}\omega^2 z + \frac{\beta_c^{(3)}}{6}\omega^3 z\right)\right)\exp\left(\mathrm{j}\,\omega t\right)\,\mathrm{d}\,\omega$$

Quantitative analysis of chromatic dispersion

Evolution of slowly varying envelope in the time domain:

$$\frac{\partial \underline{A}(z,t)}{\partial z} + \beta_c^{(1)} \frac{\partial \underline{A}(z,t)}{\partial t} - \mathrm{j}\,\frac{1}{2}\beta_c^{(2)} \frac{\partial^2 \underline{A}(z,t)}{\partial t^2} - \frac{1}{6}\beta_c^{(3)} \frac{\partial^3 \underline{A}(z,t)}{\partial t^3} = 0$$

Use retarded time frame:

$$t' = t - \beta_c^{(1)} z, \qquad z' = z, \qquad \underline{A}(z,t) = \underline{A}'\left(z, t - \beta_c^{(1)} z\right)$$

Elimination of $\beta^{(1)}$:

$$\frac{\partial \underline{A}'(z',t')}{\partial z'} - \mathrm{j}\,\frac{1}{2}\beta_c^{(2)} \frac{\partial^2 \underline{A}'(z',t')}{\partial t'^2} - \frac{1}{6}\beta_c^{(3)} \frac{\partial^3 \underline{A}'(z',t')}{\partial t'^3} = 0$$

Corresponding frequency-domain formulation for $\beta^{(3)} = 0$:

$$\underline{A}'\left(z',t'\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{\widetilde{A}}'(0,\omega)\exp\left(-\mathrm{j}\,\frac{\beta_c^{(2)}}{2}\omega^2 z'\right)\exp\left(\mathrm{j}\,\omega t'\right)\,\mathrm{d}\,\omega$$

Note that the primes are usually omitted …

Propagation of a chirped Gaussian impulse

Chirped Gaussian input impulse at z = 0 (chirp parameter α):

$$\underline{A}(0,t) = \underline{A}_o \exp\left(-\frac{(1-\mathrm{j}\,\alpha)\,t^2}{2\sigma_t^2}\right) \circ\!\!-\!\!\bullet\ \underline{\tilde{A}}(0,\omega) = \underline{A}_o\sqrt{\frac{2\pi\sigma_t^2}{(1-\mathrm{j}\,\alpha)}}\exp\left(-\frac{\sigma_t^2\omega^2}{2\,(1-\mathrm{j}\,\alpha)}\right)$$

$$= \underline{A}_o\sqrt{\frac{2\pi\sigma_t^2}{(1-\mathrm{j}\,\alpha)}}\exp\left(-\frac{(1+\mathrm{j}\,\alpha)\,\omega^2}{2\sigma_\omega^2}\right)$$

Time-bandwidth product increased by chirp:

$$\sigma_t\sigma_\omega = \sqrt{1+\alpha^2}$$

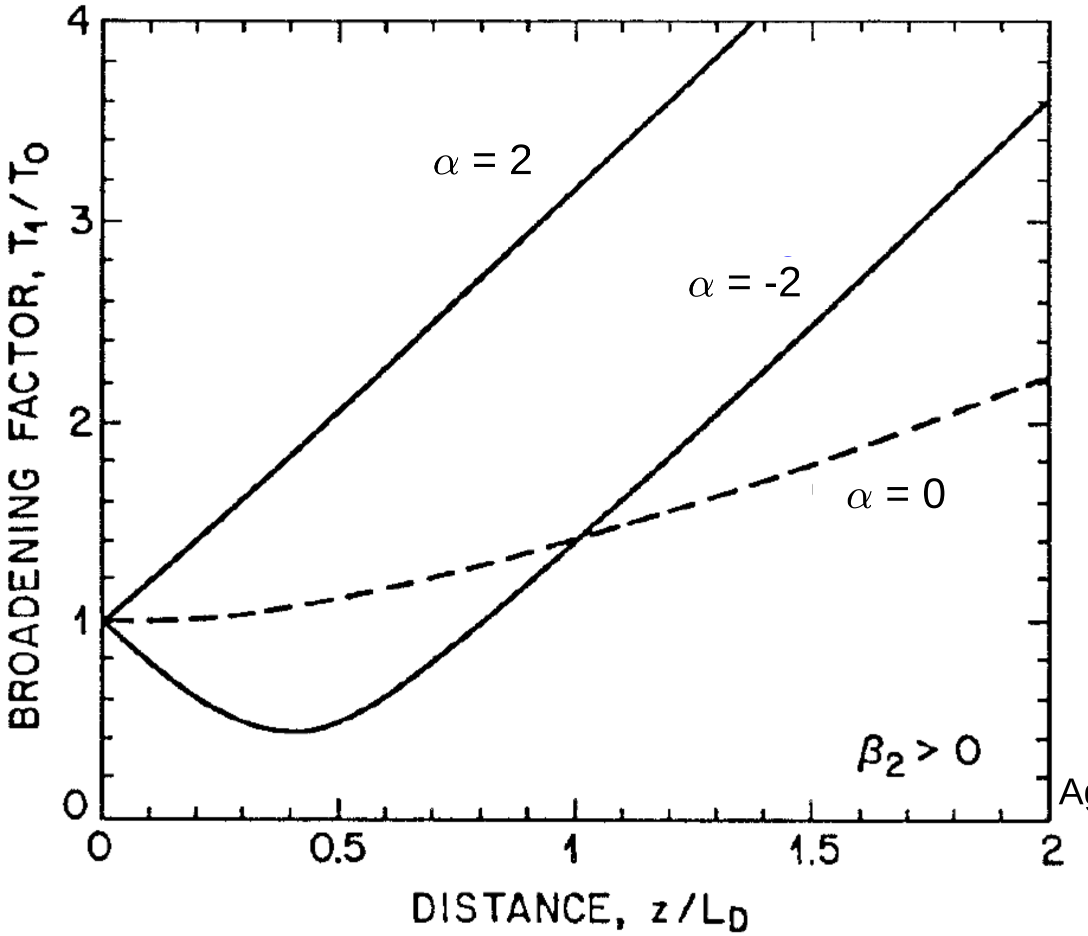
After propagation distance *z*: Gaussian shape preserved, but dispersive pulse broadening (Treatment in retarded time frame, but primes omitted!)

$$\underline{A}(z,t) = \frac{\underline{A}_o}{\sqrt{Q(z)}}\exp\left(-\frac{(1-\mathrm{j}\,\alpha)\,t^2}{2\sigma_t^2 Q(z)}\right) \qquad Q(z) = 1 + (\mathrm{j}+\alpha)\,\frac{\beta_c^{(2)} z}{\sigma_t^2}$$

Pulse broadening:

$$\frac{\sigma_t(z)}{\sigma_t(0)} = \sqrt{\left(1+\alpha\frac{\beta_c^{(2)} z}{\sigma_t^2(0)}\right)^2 + \left(\frac{\beta_c^{(2)} z}{\sigma_t^2(0)}\right)^2}$$

Gaussian impulse propagation in dispersive fibers



Agrawal, Fiber-Optic Communications Systems

- An unchirped impulse (α = 0) broadens as

$$\sqrt{1+\left(\frac{z}{L_D}\right)^2}$$

- A chirped pulse may broaden ($\beta^{(2)}\alpha > 0$) or compress ($\beta^{(2)}\alpha < 0$) during propagation, depending on the relative sign of β_2 and α

$$C_\lambda = -\frac{2\pi c}{\lambda^2}\beta_c^{(2)}$$

Dispersion length

$$L_D = \sigma_t^2(0)\,/\left|\beta_c^{(2)}\right|$$

$$\frac{\sigma_t(z)}{\sigma_t(0)} = \sqrt{\left(1+\alpha\frac{\beta_c^{(2)}z}{\sigma_t^2(0)}\right)^2 + \left(\frac{\beta_c^{(2)}z}{\sigma_t^2(0)}\right)^2}$$

Limitations of dispersive broadening on data rate

Consider narrowband optical source, modulated with unchirped Gaussian pulses:

$$\sigma_t^2(L) = \sigma_t^2(0) + \left(\frac{\beta_c^{(2)} L}{\sigma_t(0)}\right)^2$$

Spectral width of the signal defined by the modulation

Minimum width of output impulse:

$$\sigma_{t,\,\mathrm{min}}(L) = \sqrt{2\left|\beta_c^{(2)}\right| L} \quad \text{for} \quad \sigma_t(0) = \sqrt{\left|\beta_c^{(2)}\right| L}$$

Limitation on data rate *B*:

$$B\sqrt{\left|\beta_c^{(2)}\right| L} \leq \frac{1}{2\sqrt{2}}$$

$\Rightarrow$ The data rate scales with the square root of the distance

Broadband optical sources: Spectral width defined by the source

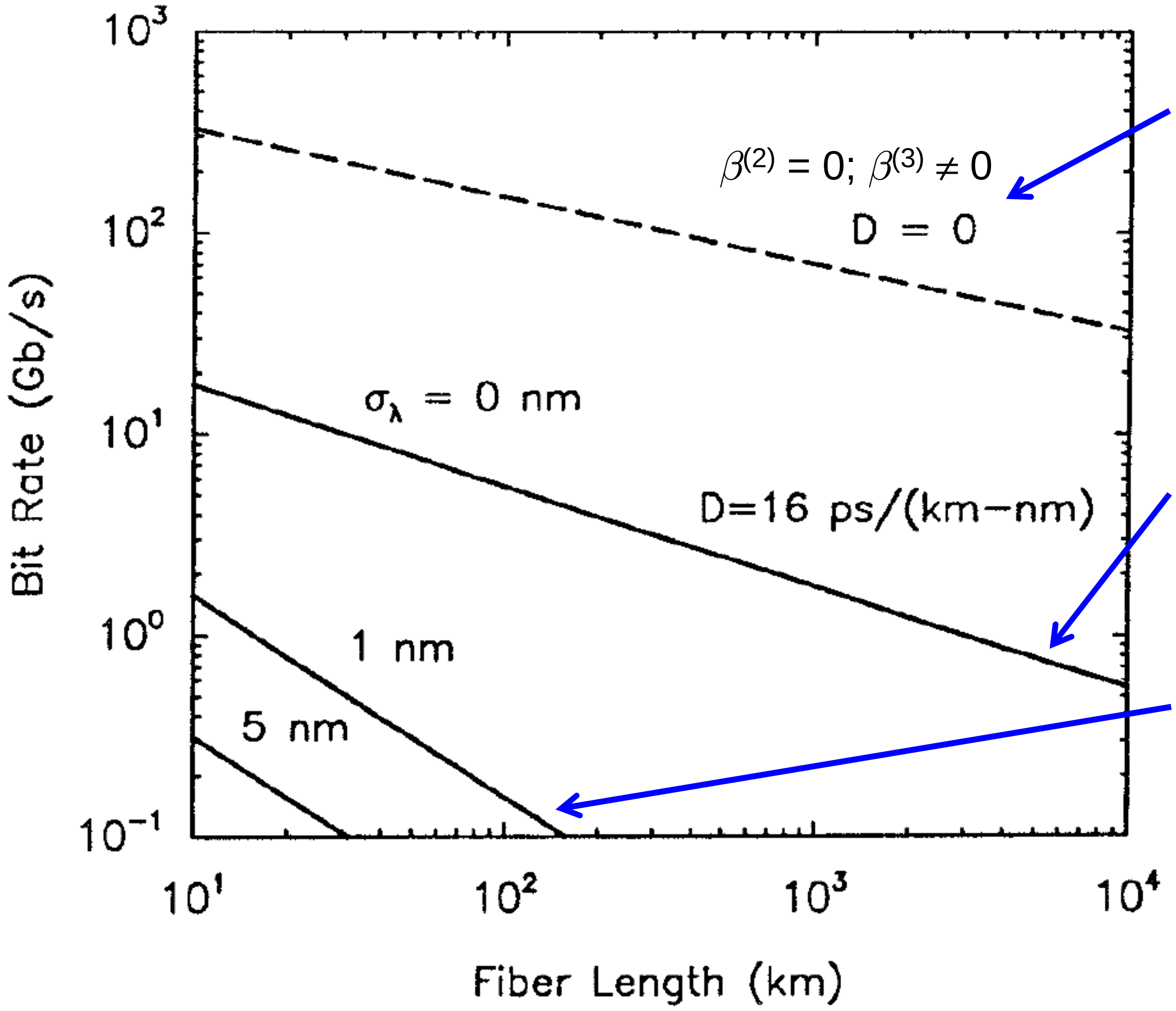
$$\sigma_t^2(L) = \sigma_t^2(0) + \left(\beta_c^{(2)} L \sigma_{\omega,\mathrm{s}}\right)^2$$

Width of the source spectrum

$$B\beta_c^{(2)} L \sigma_{\omega,\mathrm{s}} \leq \frac{1}{2}$$

$\Rightarrow$ The data rate scales linearly with the distance

Dispersion-induced bit rate limitations



Operation near zero-dispersion wavelength
⇒ Higher-order dispersion terms dominate!

Narrowband optical source:

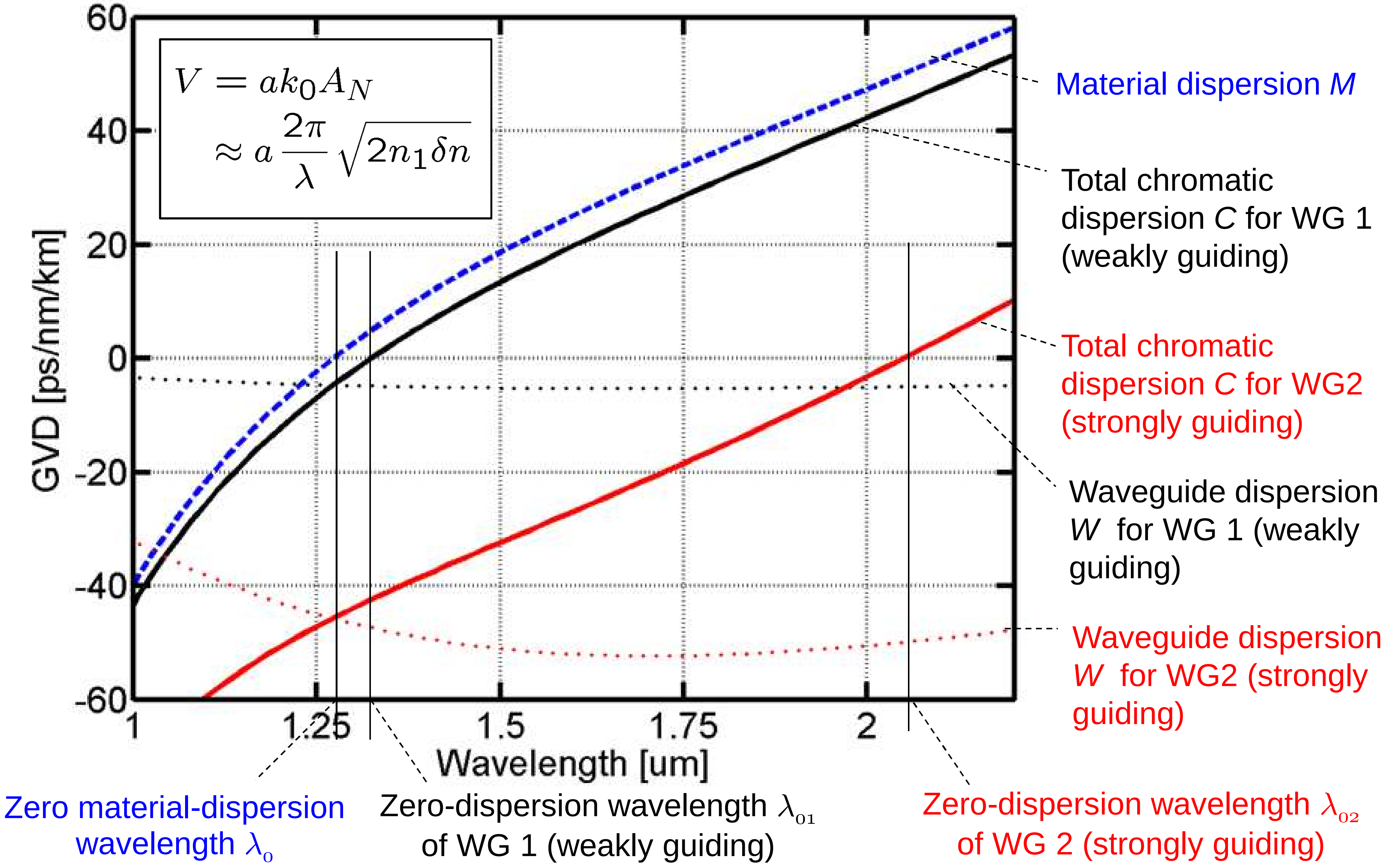
$$B \propto \frac{1}{\sqrt{L}}$$

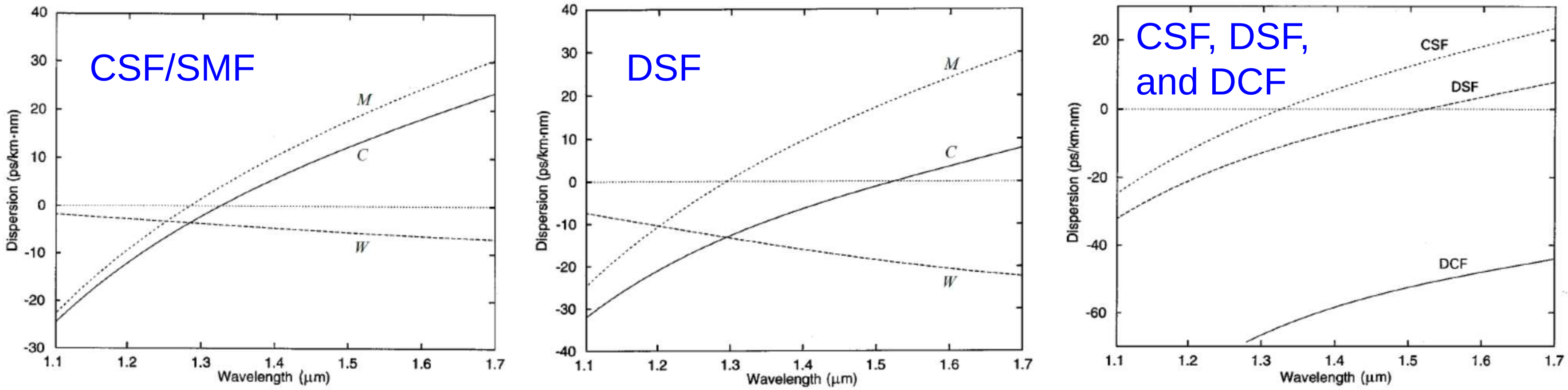
Broadband optical source:

$$B \propto \frac{1}{L}$$

Agrawal, Fiber-Optic Communications Systems

Recall: Dispersion engineering of a slab waveguide





(a) CSF: $\Delta = 0.27\,\%$, $a = 4.1\,\mu$m, $\lambda_C = 1.325\,\mu$m, $\lambda_{11G} = 1.142\,\mu$m (Table 2.1)

(b) DSF: $\Delta = 0.75\,\%$, $a = 2.3\,\mu$m, $\lambda_C = 1.523\,\mu$m, $\lambda_{11G} = 1.073\,\mu$m (Table 2.2)

(c) DCF: $\Delta = 2\,\%$, $a = 1.5\,\mu$m, $\lambda_{11G} = 1.158\,\mu$m (Table 2.3). Comparison of CSF, DCF and DCF

Dispersion Compensation:

Consider concatenated fibers (lengths L_1 and L_2) :

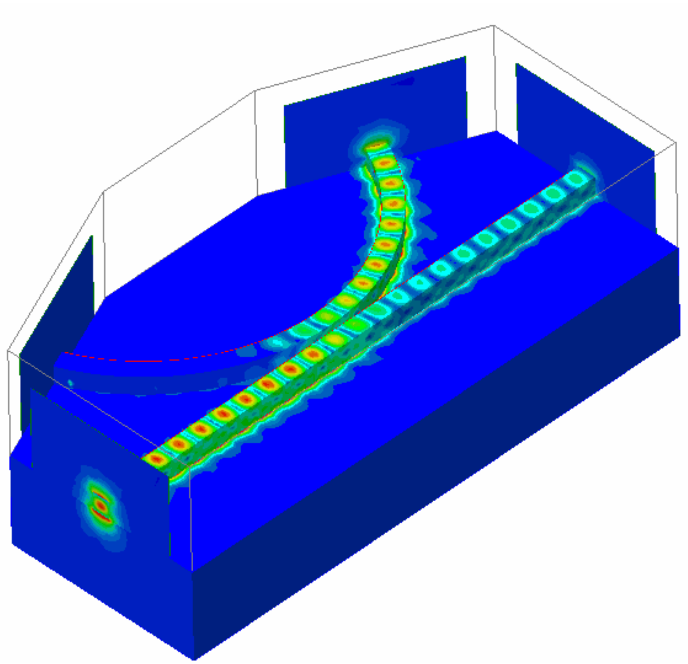
$$\underline{A}(L,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{\tilde{A}}(0,\omega)\exp\left(-\mathrm{j}\frac{1}{2}\left(\beta_{c,1}^{(2)}L_1 + \beta_{c,2}^{(2)}L_2\right)\omega^2\right)\exp(\mathrm{j}\,\omega t)\,\mathrm{d}\,\omega$$

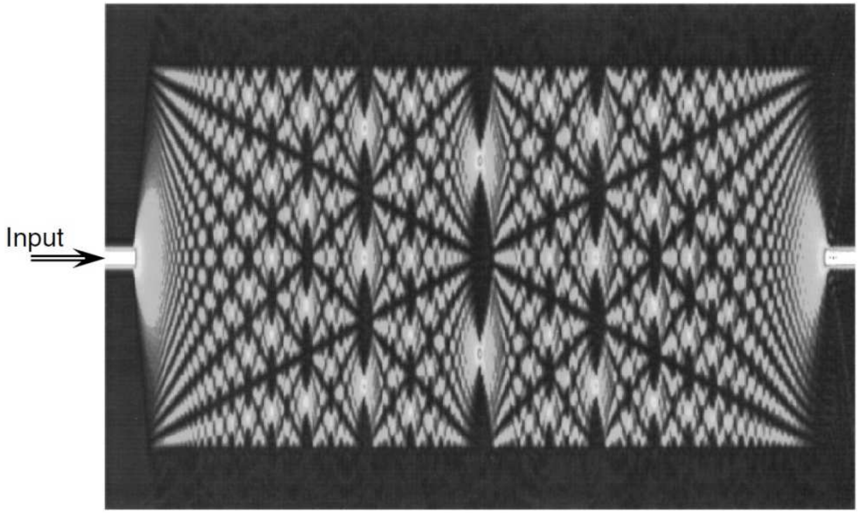
Original pulse shape reproduced for:

$$\beta_{c,1}^{(2)}L_1 + \beta_{c,2}^{(2)}L_2 = 0$$
$$C_{\lambda 1}L_1 + C_{\lambda 2}L_2 = 0$$

Dispersion compensation by using fibers with opposite signs of $\beta^{(2)}$.

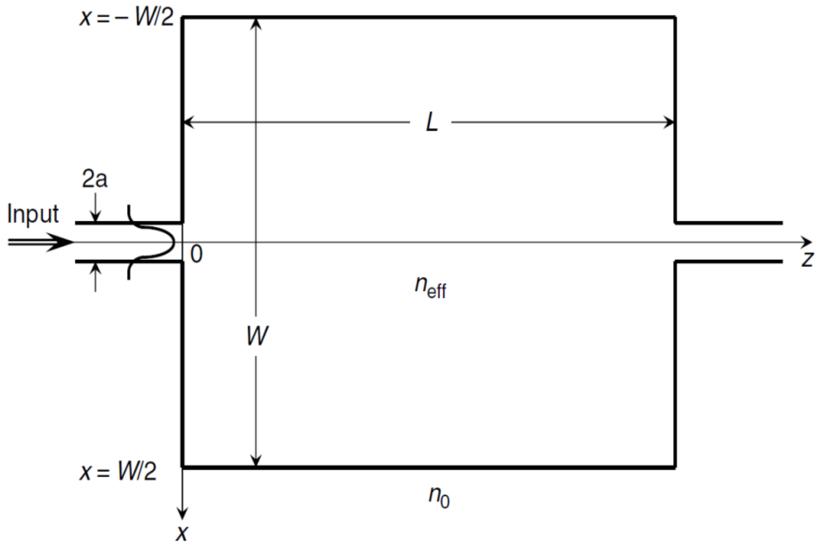
Waveguide-based devices and systems





Mode expansion method

Very often, dielectric waveguide structures are piecewise invariant in the propagation direction. For each of these sections, an **eigenmode expansion** can be used to describe the field propagation:

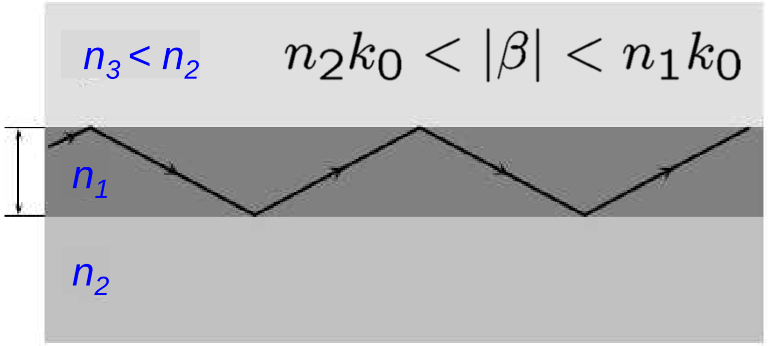


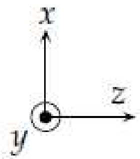
$$\underline{\mathbf{E}}(\mathbf{r},t) = \underbrace{\sum_m a_m \underline{\mathcal{E}}_m(x,y)\, \mathrm{e}^{(\mathrm{j}(\omega t - \beta_m z))}}_{\text{Guided modes}} + \underbrace{\sum_\mu \int_\rho a_\mu(\rho)\, \underline{\mathcal{E}}_{\rho,\mu}(x,y)\, \mathrm{e}^{(\mathrm{j}(\omega t - \beta_\mu(\rho) z))}\, \mathrm{d}\rho}_{\text{Radiation modes}}$$

$$\underline{\mathbf{H}}(\mathbf{r},t) = \underbrace{\sum_m a_m \underline{\mathcal{H}}_m(x,y)\, \mathrm{e}^{(\mathrm{j}(\omega t - \beta_m z))}}_{\text{Guided modes}} + \underbrace{\sum_\mu \int_\rho a_\mu(\rho)\, \underline{\mathcal{H}}_{\rho,\mu}(x,y)\, \mathrm{e}^{(\mathrm{j}(\omega t - \beta_\mu(\rho) z))}\, \mathrm{d}\rho}_{\text{Radiation modes}}$$

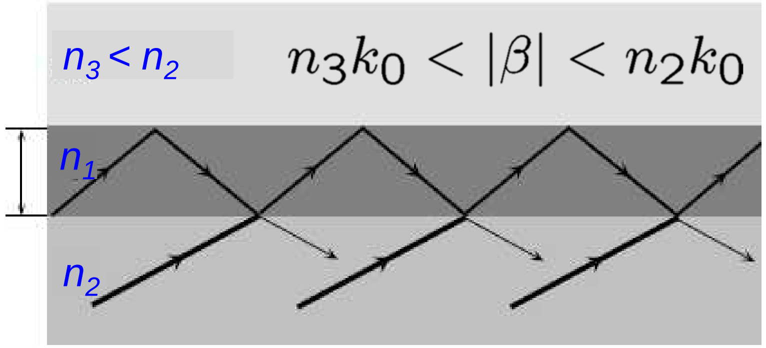
Continuous set with integration parameter ρ

Recall: Propagating eigenmodes of a slab waveguide

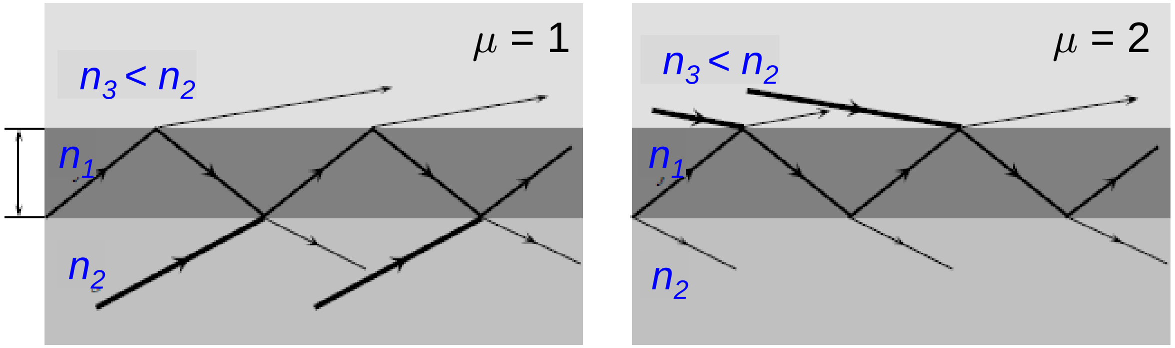




Guided mode (discrete set)



Radiation mode („substrate mode“) (continuous set)



Radiation mode („cover mode“) (continuous set)

$$|\beta| < n_3 k_0$$

Chen, Guided Wave Optics

Note: The external plane waves associated with the radiation modes can have any propagation direction.

⇒ In contrast to guided modes, propagating radiation modes form continuous sets with propagation constants

$$|\beta| < n_2 k_0 \quad \text{for} \quad \beta \in \mathbb{R} \text{ (propagating eigenmode)}$$

Completeness of mode set; scalar mode expansion

Guided modes and radiation modes form a complete basis. That means that every solution $\underline{E}(r)$, $\underline{H}(r)$ of Maxwell's equations can be represented by a superposition of these modes,

$$\underline{\mathbf{E}}(\mathbf{r}) = \sum_m a_m \underline{\mathcal{E}}_m(x,y)\,\mathrm{e}^{-\mathrm{j}\,\beta_\mu(\rho) z} + \sum_\mu \int_\rho a_\mu(\rho)\,\underline{\mathcal{E}}_{\rho,\mu}(x,y)\,\mathrm{e}^{-\mathrm{j}\,\beta_\mu(\rho) z}\,\mathrm{d}\rho$$

$$\underline{\mathbf{H}}(\mathbf{r}) = \sum_m a_m \underline{\mathcal{H}}_m(x,y)\,\mathrm{e}^{-\mathrm{j}\,\beta_\mu(\rho) z} + \sum_\mu \int_\rho a_\mu(\rho)\,\underline{\mathcal{H}}_{\rho,\mu}(x,y)\,\mathrm{e}^{-\mathrm{j}\,\beta_\mu(\rho) z}\,\mathrm{d}\rho$$

For modes of weakly guiding low index-contrast waveguides, one transverse field component is usually much stronger than the other field components. For a given polarization, the dominant transverse component can then be associated with a scalar function Ψ(x,y,), and the vectorial mode expansion can be reduced to a scalar expansion of the dominant transverse field component Φ(x,y)

$$\Phi(x,y,z) = \sum_m a_m \underline{\Psi}_m(x,y)\,\mathrm{e}^{-\mathrm{j}\,\beta_m z} + \sum_\mu \int_\rho a_\mu(\rho)\,\underline{\Psi}_{\rho,\mu}(x,y) e^{-\mathrm{j}\,\beta_\mu(\rho) z}\,\mathrm{d}\rho$$

Note that this expansion is not any more complete (valid for one polarization only!)

Orthogonality relations of mode fields

Consider two guided waveguide modes:

Vector differential operators

$$\underline{\mathbf{E}}_\nu(x,y,z) = \underline{\mathcal{E}}_\nu(x,y)\exp\left(-\mathrm{j}\,\beta_\nu z\right),$$
$$\underline{\mathbf{H}}_\mu(x,y,z) = \underline{\mathcal{H}}_\mu(x,y)\exp\left(-\mathrm{j}\,\beta_\mu z\right)$$

⇒ Orthogonality relation for guided modes:

$$\frac{1}{4}\iint_{-\infty}^{\infty}\left(\underline{\mathcal{E}}_\nu(x,y)\times\underline{\mathcal{H}}_\mu^\star(x,y)+\underline{\mathcal{E}}_\mu^\star(x,y)\times\underline{\mathcal{H}}_\nu(x,y)\right)\cdot\mathbf{e}_z\,\mathrm{d}\,x\,\mathrm{d}\,y = \mathcal{P}_\mu\delta_{\nu\mu}$$

where $$\mathcal{P}_\mu = \frac{1}{2}\iint_{-\infty}^{\infty}\mathrm{Re}\left\{\underline{\mathcal{E}}_\mu(x,y)\times\underline{\mathcal{H}}_\mu^\star(x,y)\right\}\cdot\mathbf{e}_z\,\mathrm{d}\,x\,\mathrm{d}\,y$$

Similarly: Orthogonality relation for radiation modes

$$\frac{1}{4}\iint_{-\infty}^{\infty}\left(\underline{\mathcal{E}}_{\rho,\mu}(x,y)\times\underline{\mathcal{H}}_{\rho',\mu'}^\star(x,y)+\underline{\mathcal{E}}_{\rho',\mu'}^\star(x,y)\times\underline{\mathcal{H}}_{\rho,\mu}(x,y)\right)\cdot\mathbf{e}_z\,\mathrm{d}\,x\,\mathrm{d}\,y = \mathcal{P}_{\rho,\mu}\delta_{\mu\mu'}\delta\left(\rho-\rho'\right)$$

where $$\frac{1}{2}\iint_{-\infty}^{\infty}\mathrm{Re}\left\{\underline{\mathcal{E}}_{\rho,\mu}(x,y)\times\underline{\mathcal{H}}_{\rho',\mu}^\star(x,y)\right\}\cdot\mathbf{e}_z\,\mathrm{d}\,x\,\mathrm{d}\,y = \mathcal{P}_{\rho,\mu}\delta\left(\rho-\rho'\right)$$

Guided modes and radiation modes are always orthogonal to each other:

$$\frac{1}{4}\iint_{-\infty}^{\infty}\left(\underline{\mathcal{E}}_\nu(x,y)\times\underline{\mathcal{H}}_{\rho,\mu}^\star(x,y)+\underline{\mathcal{E}}_{\rho,\mu}^\star(x,y)\times\underline{\mathcal{H}}_\nu(x,y)\right)\cdot\mathbf{e}_z\,\mathrm{d}\,x\,\mathrm{d}\,y = 0$$

Vector differential operators

Linearität	
1. $\nabla(\alpha\Phi+\beta\Psi)=\alpha\,\nabla\Phi+\beta\,\nabla\Psi$	$\mathrm{grad}(\alpha\Phi+\beta\Psi)=\alpha\,\mathrm{grad}\,\Phi+\beta\,\mathrm{grad}\,\Psi$
2. $\nabla\cdot(\alpha\mathbf{F}+\beta\mathbf{G})=\alpha\,\nabla\cdot\mathbf{F}+\beta\,\nabla\cdot\mathbf{G}$	$\mathrm{div}(\alpha\mathbf{F}+\beta\mathbf{G})=\alpha\,\mathrm{div}\,\mathbf{F}+\beta\,\mathrm{div}\,\mathbf{G}$
3. $\nabla\times(\alpha\mathbf{F}+\beta\mathbf{G})=\alpha\,\nabla\times\mathbf{F}+\beta\,\nabla\times\mathbf{G}$	$\mathrm{rot}(\alpha\mathbf{F}+\beta\mathbf{G})=\alpha\,\mathrm{rot}\,\mathbf{F}+\beta\,\mathrm{rot}\,\mathbf{G}$
Operation auf Produkten	
4. $\nabla(\Phi\Psi)=\Phi\,\nabla\Psi+\Psi\,\nabla\Phi$	$\mathrm{grad}(\Phi\Psi)=\Phi\,\mathrm{grad}\,\Psi+\Psi\,\mathrm{grad}\,\Phi$
5. $\nabla(\mathbf{F}\cdot\mathbf{G})=(\mathbf{F}\cdot\nabla)\mathbf{G}+(\mathbf{G}\cdot\nabla)\mathbf{F}+$ $+\mathbf{F}\times(\nabla\times\mathbf{G})+\mathbf{G}\times(\nabla\times\mathbf{F})$	$\mathrm{grad}(\mathbf{F}\cdot\mathbf{G})=(\mathbf{F}\cdot\mathrm{grad})\mathbf{G}+$ $+(\mathbf{G}\cdot\mathrm{grad})\mathbf{F}+\mathbf{F}\times\mathrm{rot}\,\mathbf{G}+\mathbf{G}\times\mathrm{rot}\,\mathbf{F}$
6. $\nabla\cdot(\Phi\mathbf{F})=\Phi\,\nabla\cdot\mathbf{F}+(\nabla\Phi)\cdot\mathbf{F}$	$\mathrm{div}(\Phi\mathbf{F})=\Phi\,\mathrm{div}\,\mathbf{F}+\mathbf{F}\cdot\mathrm{grad}\,\Phi$
7. $\nabla\cdot(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot\nabla\times\mathbf{F}-\mathbf{F}\cdot\nabla\times\mathbf{G}$	$\mathrm{div}(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot\mathrm{rot}\,\mathbf{F}-\mathbf{F}\cdot\mathrm{rot}\,\mathbf{G}$
8. $\nabla\times(\Phi\mathbf{F})=\Phi\,\nabla\times\mathbf{F}+(\nabla\Phi)\times\mathbf{F}$	$\mathrm{rot}(\Phi\mathbf{F})=\Phi\,\mathrm{rot}\,\mathbf{F}+(\mathrm{grad}\,\Phi)\times\mathbf{F}$
9. $\nabla\times(\mathbf{F}\times\mathbf{G})=(\mathbf{G}\cdot\nabla)\mathbf{F}-(\mathbf{F}\cdot\nabla)\mathbf{G}+$ $+\mathbf{F}(\nabla\cdot\mathbf{G})-\mathbf{G}(\nabla\cdot\mathbf{F})$	$\mathrm{rot}(\mathbf{F}\times\mathbf{G})=(\mathbf{G}\cdot\mathrm{grad})\mathbf{F}-$ $-(\mathbf{F}\cdot\mathrm{grad})\mathbf{G}+\mathbf{F}\,\mathrm{div}\,\mathbf{G}-\mathbf{G}\,\mathrm{div}\,\mathbf{F}$
Zweifache Anwendung von ∇	
10. $\nabla\cdot(\nabla\times\mathbf{F})=0$	$\mathrm{div}\,\mathrm{rot}\,\mathbf{F}=0$
11. $\nabla\times(\nabla\Phi)=\mathbf{0}$	$\mathrm{rot}\,\mathrm{grad}\,\Phi=\mathbf{0}$
12. $\nabla\times(\nabla\times\mathbf{F})=\nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F}$	$\mathrm{rot}\,\mathrm{rot}\,\mathbf{F}=\mathrm{grad}\,\mathrm{div}\,\mathbf{F}-\Delta\mathbf{F}$

Back to orthogonality relations …

Simplified orthogonality relations

Maxwell's equations for guided modes are invariant under the transformation:

$$\beta \to -\beta$$
$$\mathcal{E}_x \to \mathcal{E}_x$$
$$\mathcal{E}_y \to \mathcal{E}_y$$
$$\mathcal{E}_z \to -\mathcal{E}_z$$
$$\mathcal{H}_x \to -\mathcal{H}_x$$
$$\mathcal{H}_y \to -\mathcal{H}_y$$
$$\mathcal{H}_z \to \mathcal{H}_z$$

$$\frac{\partial \underline{\mathcal{E}}_z}{\partial y} + \mathrm{j}\,\beta \underline{\mathcal{E}}_y = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_x \qquad \frac{\partial \underline{\mathcal{H}}_z}{\partial y} + \mathrm{j}\,\beta \underline{\mathcal{H}}_y = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_x$$

$$-\frac{\partial \underline{\mathcal{E}}_z}{\partial x} - \mathrm{j}\,\beta \underline{\mathcal{E}}_x = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_y \qquad -\frac{\partial \underline{\mathcal{H}}_z}{\partial x} - \mathrm{j}\,\beta \underline{\mathcal{H}}_x = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_y$$

$$\frac{\partial \underline{\mathcal{E}}_y}{\partial x} - \frac{\partial \underline{\mathcal{E}}_x}{\partial y} = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_z \qquad \frac{\partial \underline{\mathcal{H}}_y}{\partial x} - \frac{\partial \underline{\mathcal{H}}_x}{\partial y} = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_z$$

=> Simplification for waveguide modes propagating in the same direction ($\beta_\nu \neq -\beta_\mu$):

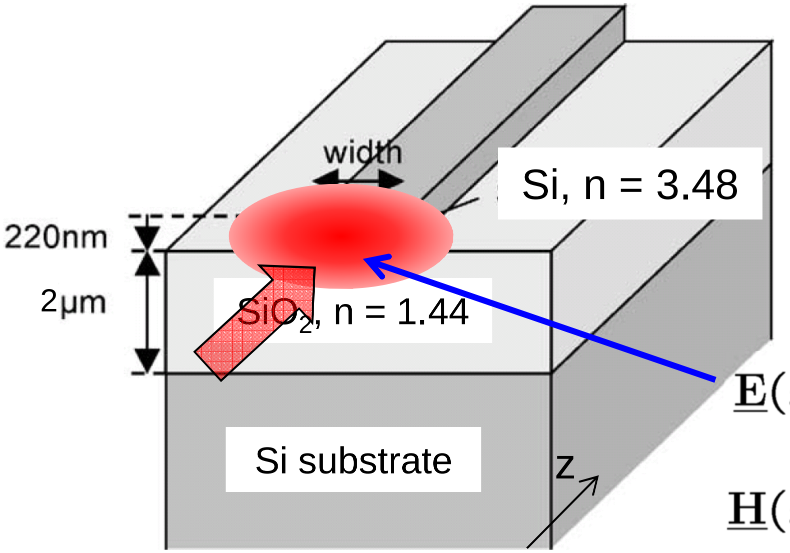
$$\frac{1}{2}\iint_{-\infty}^{\infty} \mathrm{Re}\left\{\underline{\mathcal{E}}_\nu(x,y) \times \underline{\mathcal{H}}_\mu^\star(x,y)\right\} \cdot \mathbf{e}_z \,\mathrm{d}\,x\,\mathrm{d}\,y = \mathcal{P}_\mu \delta_{\nu\mu}$$

For scalar mode field representations of weakly guiding, low-index contrast waveguides:

$$\frac{\beta_\nu}{2\omega\mu_0}\iint_{-\infty}^{\infty} \underline{\Psi}_\nu(x,y)\,\underline{\Psi}_\mu^\star(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y = \mathcal{P}_\mu \delta_{\nu\mu}$$

where $$\mathcal{P}_\mu = \frac{\beta_\mu}{2\omega\mu_0}\iint_{-\infty}^{\infty} |\underline{\Psi}_\mu(x,y)|^2 \,\mathrm{d}\,x\,\mathrm{d}\,y$$

Example: Coupling efficiency



Illumination of a waveguide facet with a free-space beam
=> How much power is coupled into the fundamental waveguide mode?

Mode expansion of illuminating field at z = 0:

$$\underline{\mathbf{E}}(x,y,0) = \sum_m a_m \underline{\mathcal{E}}_m(x,y) + \sum_\mu \int_\rho a_\mu(\rho)\,\underline{\mathcal{E}}_{\rho,\mu}(x,y)\,\mathrm{d}\rho$$

$$\underline{\mathbf{H}}(x,y,0) = \sum_m a_m \underline{\mathcal{H}}_m(x,y) + \sum_\mu \int_\rho a_\mu(\rho)\,\underline{\mathcal{H}}_{\rho,\mu}(x,y)\,\mathrm{d}\rho$$

Fundamental mode amplitude:

$$a_0 = \frac{1}{4\mathcal{P}_0} \cdot \iint_{-\infty}^{\infty} \left(\underline{\mathbf{E}}(x,y,0) \times \underline{\mathcal{H}}_0^\star(x,y) + \underline{\mathcal{E}}_0^\star(x,y) \times \underline{\mathbf{H}}(x,y,0)\right) \cdot \mathbf{e}_z\,\mathrm{d}x\,\mathrm{d}y$$

Power coupling efficiency:

$$\eta = \frac{\left|\frac{1}{4} \cdot \iint_{-\infty}^{\infty} \left(\underline{\mathbf{E}}(x,y,0) \times \underline{\mathcal{H}}_0^\star(x,y) + \underline{\mathcal{E}}_0^\star(x,y) \times \underline{\mathbf{H}}(x,y,0)\right) \cdot \mathbf{e}_z\,\mathrm{d}x\,\mathrm{d}y\right|^2}{\frac{1}{2}\iint_{-\infty}^{\infty} \mathrm{Re}\left\{\underline{\mathcal{E}}_0(x,y) \times \underline{\mathcal{H}}_0^\star(x,y)\right\} \cdot \mathbf{e}_z\,\mathrm{d}x\,\mathrm{d}y \cdot \frac{1}{2}\iint_{-\infty}^{\infty} \mathrm{Re}\left\{\underline{\mathbf{E}}(x,y,0) \times \underline{\mathbf{H}}^\star(x,y,0)\right\} \cdot \mathbf{e}_z\,\mathrm{d}x\,\mathrm{d}y}$$

For scalar mode fields (problem set!):

$$\eta = \frac{\left|\iint_{-\infty}^{\infty} \underline{\Phi}(x,y,0)\,\underline{\Psi}_0^\star(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|^2}{\iint_{-\infty}^{\infty} |\underline{\Psi}_0(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \cdot \iint_{-\infty}^{\infty} |\underline{\Phi}(x,y,0)|^2\,\mathrm{d}x\,\mathrm{d}y}$$

Orthogonality relations of mode fields

Consider two guided waveguide modes:

[Vector differential operators](#)

$$\underline{\mathbf{E}}_\nu(x,y,z) = \underline{\mathcal{E}}_\nu(x,y)\exp\left(-\mathrm{j}\,\beta_\nu z\right),$$
$$\underline{\mathbf{H}}_\mu(x,y,z) = \underline{\mathcal{H}}_\mu(x,y)\exp\left(-\mathrm{j}\,\beta_\mu z\right)$$

$\Rightarrow$ Orthogonality relation for guided modes:

$$\frac{1}{4}\iint_{-\infty}^{\infty}\left(\underline{\mathcal{E}}_\nu(x,y)\times\underline{\mathcal{H}}_\mu^\star(x,y)+\underline{\mathcal{E}}_\mu^\star(x,y)\times\underline{\mathcal{H}}_\nu(x,y)\right)\cdot\mathbf{e}_z\,\mathrm{d}\,x\,\mathrm{d}\,y = \mathcal{P}_\mu\delta_{\nu\mu}$$

where $\mathcal{P}_\mu = \frac{1}{2}\iint_{-\infty}^{\infty}\mathrm{Re}\left\{\underline{\mathcal{E}}_\mu(x,y)\times\underline{\mathcal{H}}_\mu^\star(x,y)\right\}\cdot\mathbf{e}_z\,\mathrm{d}\,x\,\mathrm{d}\,y$

Similarly: Orthogonality relation for radiation modes

$$\frac{1}{4}\iint_{-\infty}^{\infty}\left(\underline{\mathcal{E}}_{\rho,\mu}(x,y)\times\underline{\mathcal{H}}_{\rho',\mu'}^\star(x,y)+\underline{\mathcal{E}}_{\rho',\mu'}^\star(x,y)\times\underline{\mathcal{H}}_{\rho,\mu}(x,y)\right)\cdot\mathbf{e}_z\,\mathrm{d}\,x\,\mathrm{d}\,y = \mathcal{P}_{\rho,\mu}\delta_{\mu\mu'}\delta\left(\rho-\rho'\right)$$

where $\frac{1}{2}\iint_{-\infty}^{\infty}\mathrm{Re}\left\{\underline{\mathcal{E}}_{\rho,\mu}(x,y)\times\underline{\mathcal{H}}_{\rho',\mu}^\star(x,y)\right\}\cdot\mathbf{e}_z\,\mathrm{d}\,x\,\mathrm{d}\,y = \mathcal{P}_{\rho,\mu}\delta\left(\rho-\rho'\right)$

Guided modes and radiation modes are always orthogonal to each other:

$$\frac{1}{4}\iint_{-\infty}^{\infty}\left(\underline{\mathcal{E}}_\nu(x,y)\times\underline{\mathcal{H}}_{\rho,\mu}^\star(x,y)+\underline{\mathcal{E}}_{\rho,\mu}^\star(x,y)\times\underline{\mathcal{H}}_\nu(x,y)\right)\cdot\mathbf{e}_z\,\mathrm{d}\,x\,\mathrm{d}\,y = 0$$

Vector differential operators

Linearität	
1. $\nabla(\alpha\Phi+\beta\Psi)=\alpha\nabla\Phi+\beta\nabla\Psi$	$\mathrm{grad}(\alpha\Phi+\beta\Psi)=\alpha\,\mathrm{grad}\,\Phi+\beta\,\mathrm{grad}\,\Psi$
2. $\nabla\cdot(\alpha\mathbf{F}+\beta\mathbf{G})=\alpha\nabla\cdot\mathbf{F}+\beta\nabla\cdot\mathbf{G}$	$\mathrm{div}(\alpha\mathbf{F}+\beta\mathbf{G})=\alpha\,\mathrm{div}\,\mathbf{F}+\beta\,\mathrm{div}\,\mathbf{G}$
3. $\nabla\times(\alpha\mathbf{F}+\beta\mathbf{G})=\alpha\nabla\times\mathbf{F}+\beta\nabla\times\mathbf{G}$	$\mathrm{rot}(\alpha\mathbf{F}+\beta\mathbf{G})=\alpha\,\mathrm{rot}\,\mathbf{F}+\beta\,\mathrm{rot}\,\mathbf{G}$
Operation auf Produkten	
4. $\nabla(\Phi\Psi)=\Phi\nabla\Psi+\Psi\nabla\Phi$	$\mathrm{grad}(\Phi\Psi)=\Phi\,\mathrm{grad}\,\Psi+\Psi\,\mathrm{grad}\,\Phi$
5. $\nabla(\mathbf{F}\cdot\mathbf{G})=(\mathbf{F}\cdot\nabla)\mathbf{G}+(\mathbf{G}\cdot\nabla)\mathbf{F}+\mathbf{F}\times(\nabla\times\mathbf{G})+\mathbf{G}\times(\nabla\times\mathbf{F})$	$\mathrm{grad}(\mathbf{F}\cdot\mathbf{G})=(\mathbf{F}\cdot\mathrm{grad})\mathbf{G}+(\mathbf{G}\cdot\mathrm{grad})\mathbf{F}+\mathbf{F}\times\mathrm{rot}\,\mathbf{G}+\mathbf{G}\times\mathrm{rot}\,\mathbf{F}$
6. $\nabla\cdot(\Phi\mathbf{F})=\Phi\nabla\cdot\mathbf{F}+(\nabla\Phi)\cdot\mathbf{F}$	$\mathrm{div}(\Phi\mathbf{F})=\Phi\,\mathrm{div}\,\mathbf{F}+\mathbf{F}\cdot\mathrm{grad}\,\Phi$
7. $\nabla\cdot(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot\nabla\times\mathbf{F}-\mathbf{F}\cdot\nabla\times\mathbf{G}$	$\mathrm{div}(\mathbf{F}\times\mathbf{G})=\mathbf{G}\cdot\mathrm{rot}\,\mathbf{F}-\mathbf{F}\cdot\mathrm{rot}\,\mathbf{G}$
8. $\nabla\times(\Phi\mathbf{F})=\Phi\nabla\times\mathbf{F}+(\nabla\Phi)\times\mathbf{F}$	$\mathrm{rot}(\Phi\mathbf{F})=\Phi\,\mathrm{rot}\,\mathbf{F}+(\mathrm{grad}\,\Phi)\times\mathbf{F}$
9. $\nabla\times(\mathbf{F}\times\mathbf{G})=(\mathbf{G}\cdot\nabla)\mathbf{F}-(\mathbf{F}\cdot\nabla)\mathbf{G}+\mathbf{F}(\nabla\cdot\mathbf{G})-\mathbf{G}(\nabla\cdot\mathbf{F})$	$\mathrm{rot}(\mathbf{F}\times\mathbf{G})=(\mathbf{G}\cdot\mathrm{grad})\mathbf{F}-(\mathbf{F}\cdot\mathrm{grad})\mathbf{G}+\mathbf{F}\,\mathrm{div}\,\mathbf{G}-\mathbf{G}\,\mathrm{div}\,\mathbf{F}$
Zweifache Anwendung von ∇	
10. $\nabla\cdot(\nabla\times\mathbf{F})=0$	$\mathrm{div}\,\mathrm{rot}\,\mathbf{F}=0$
11. $\nabla\times(\nabla\Phi)=\mathbf{0}$	$\mathrm{rot}\,\mathrm{grad}\,\Phi=\mathbf{0}$
12. $\nabla\times(\nabla\times\mathbf{F})=\nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F}$	$\mathrm{rot}\,\mathrm{rot}\,\mathbf{F}=\mathrm{grad}\,\mathrm{div}\,\mathbf{F}-\Delta\mathbf{F}$

Back to orthogonality relations …

Simplified orthogonality relations

Maxwell's equations for guided modes are invariant under the transformation:

$$\beta \rightarrow -\beta$$
$$\mathcal{E}_x \rightarrow \mathcal{E}_x$$
$$\mathcal{E}_y \rightarrow \mathcal{E}_y$$
$$\mathcal{E}_z \rightarrow -\mathcal{E}_z$$
$$\mathcal{H}_x \rightarrow -\mathcal{H}_x$$
$$\mathcal{H}_y \rightarrow -\mathcal{H}_y$$
$$\mathcal{H}_z \rightarrow \mathcal{H}_z$$

$$\frac{\partial \underline{\mathcal{E}}_z}{\partial y} + \mathrm{j}\,\beta \underline{\mathcal{E}}_y = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_x \qquad \frac{\partial \underline{\mathcal{H}}_z}{\partial y} + \mathrm{j}\,\beta \underline{\mathcal{H}}_y = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_x$$

$$-\frac{\partial \underline{\mathcal{E}}_z}{\partial x} - \mathrm{j}\,\beta \underline{\mathcal{E}}_x = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_y \qquad -\frac{\partial \underline{\mathcal{H}}_z}{\partial x} - \mathrm{j}\,\beta \underline{\mathcal{H}}_x = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_y$$

$$\frac{\partial \underline{\mathcal{E}}_y}{\partial x} - \frac{\partial \underline{\mathcal{E}}_x}{\partial y} = -\mathrm{j}\,\omega\mu_0 \underline{\mathcal{H}}_z \qquad \frac{\partial \underline{\mathcal{H}}_y}{\partial x} - \frac{\partial \underline{\mathcal{H}}_x}{\partial y} = \mathrm{j}\,\omega\epsilon_0 n^2 \underline{\mathcal{E}}_z$$

=> Simplification for waveguide modes propagating in the same direction ($\beta_\nu \neq -\beta_\mu$):

$$\frac{1}{2}\iint_{-\infty}^{\infty} \mathrm{Re}\left\{\underline{\mathcal{E}}_\nu(x,y) \times \underline{\mathcal{H}}_\mu^\star(x,y)\right\} \cdot \mathbf{e}_z \,\mathrm{d}\,x\,\mathrm{d}\,y = \mathcal{P}_\mu \delta_{\nu\mu}$$

For scalar mode field representations of weakly guiding, low-index contrast waveguides:

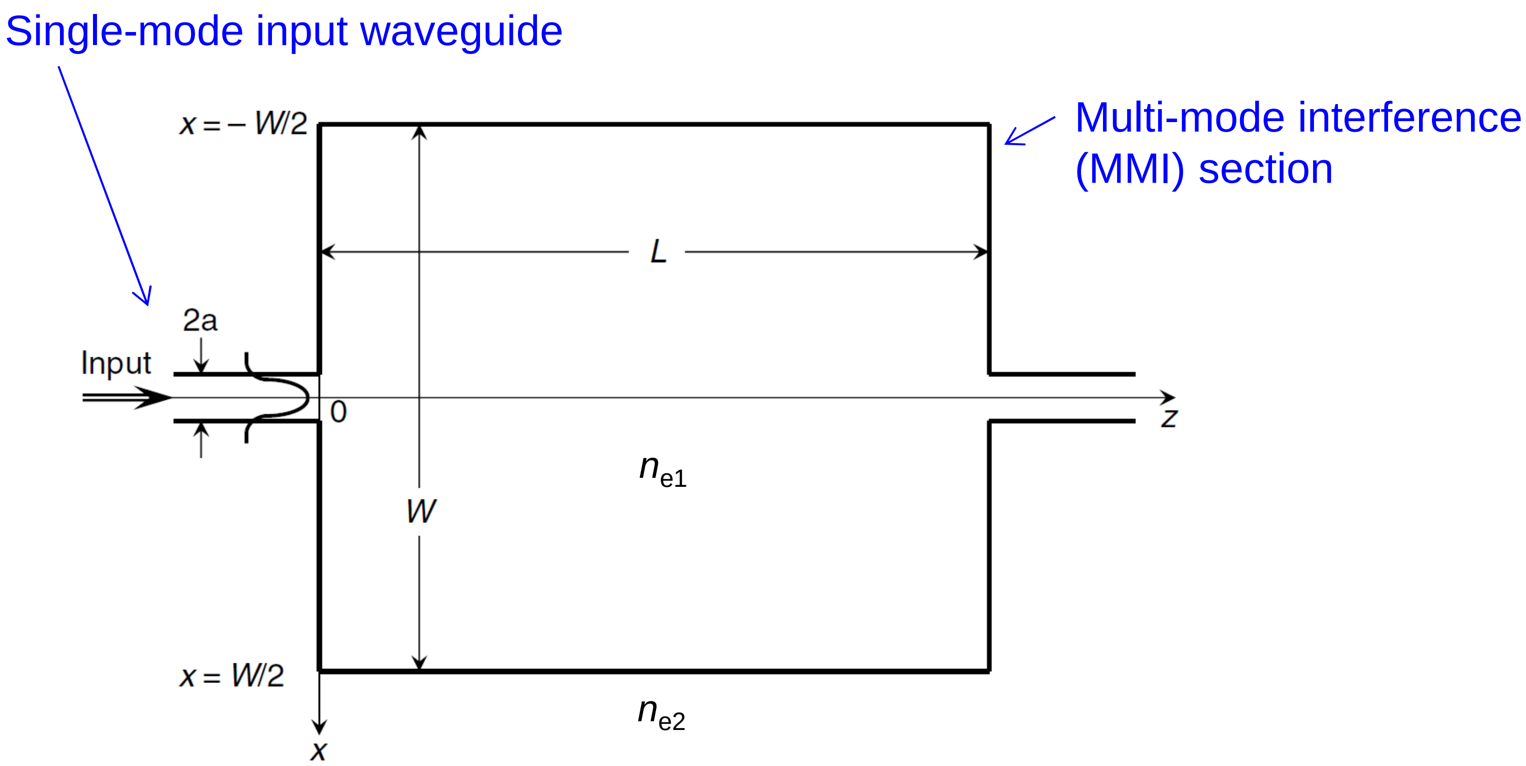
$$\frac{\beta_\nu}{2\omega\mu_0}\iint_{-\infty}^{\infty} \underline{\Psi}_\nu(x,y)\,\underline{\Psi}_\mu^\star(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y = \mathcal{P}_\mu \delta_{\nu\mu}$$

where

$$\mathcal{P}_\mu = \frac{\beta_\mu}{2\omega\mu_0}\iint_{-\infty}^{\infty} |\underline{\Psi}_\mu(x,y)|^2\,\mathrm{d}\,x\,\mathrm{d}\,y$$

Multi-mode interference (MMI) coupler

Basic configuration:



$$\Phi(x,y,0) = \sum_m a_m \underline{\Psi}_m(x,y)$$

where $$a_m = \frac{\beta}{2\mathcal{P}_m \omega \mu_0} \iint_{-\infty}^{\infty} \Phi(x,y,0)\, \underline{\Psi}_m^{\star}(x,y)\, \mathrm{d}\,x\, \mathrm{d}\,y$$

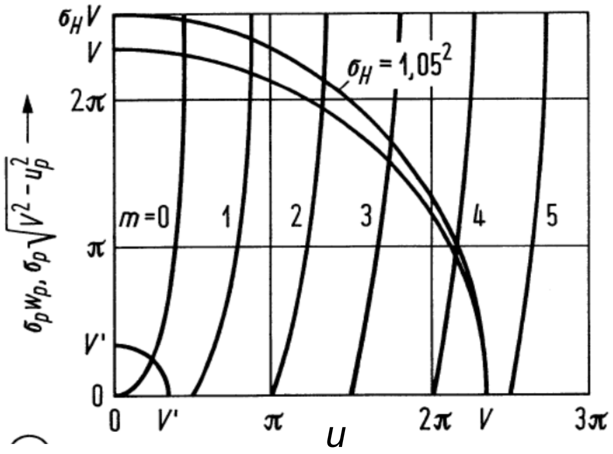
Okamoto, Fundamentals of Optical Waveguides

Analysis of MMI section

Propagation in MMI section:

$$\Phi(x,y,z) = \sum_m a_m \underline{\Psi}_m(x,y)\,\mathrm{e}^{-\mathrm{j}\,\beta_m z}$$

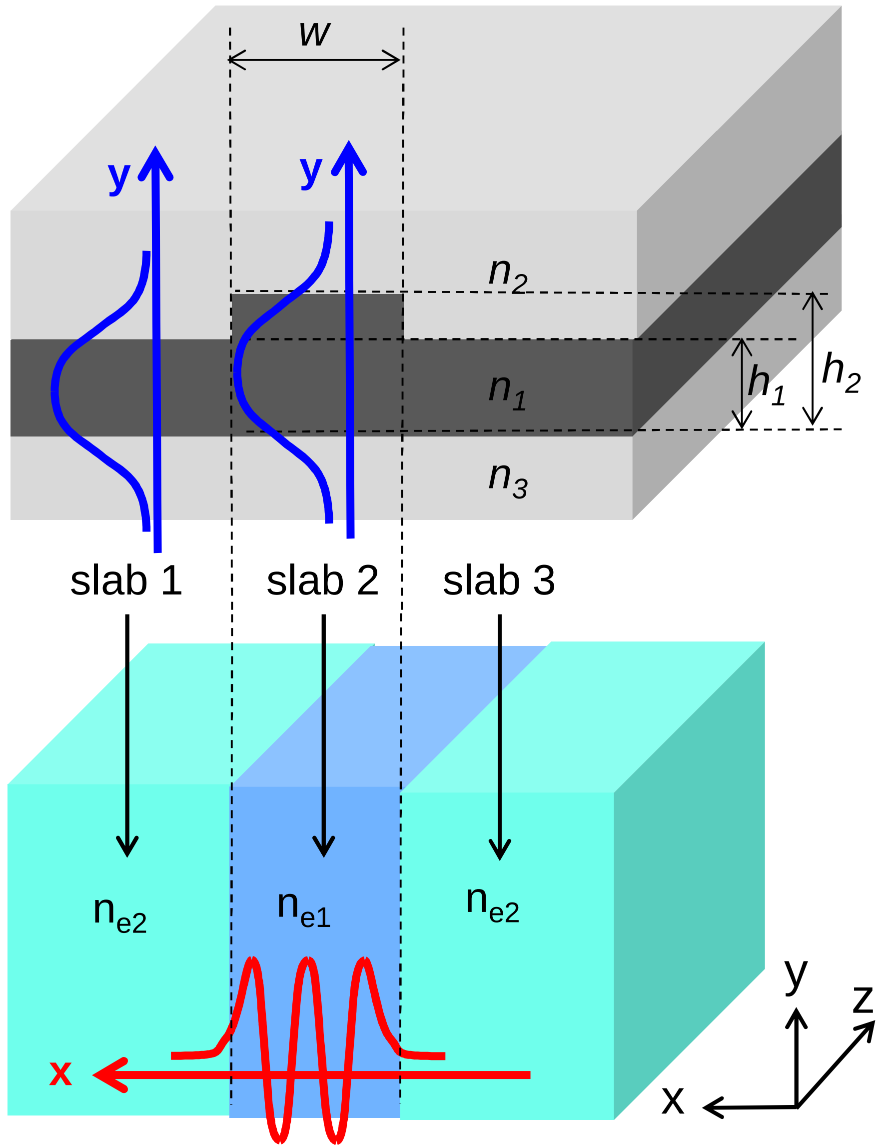
Reduce 3D problem to 2D by using the effective-index approximation…



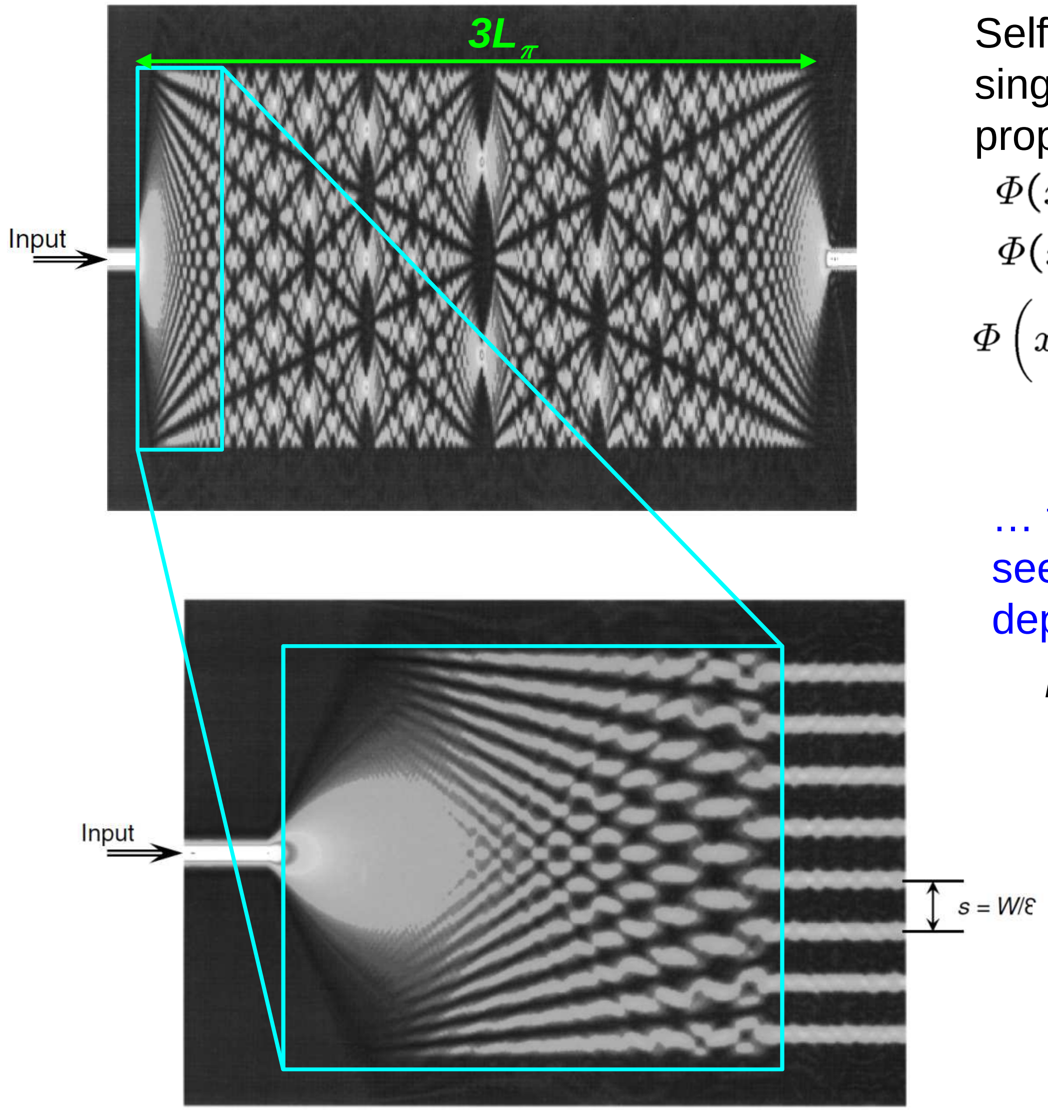
Far from cut-off, we find: $u_m \approx (m+1)\,\dfrac{\pi}{2}$

$$\beta_0 - \beta_m = \frac{m\,(m+2)\,\pi}{3L_\pi}$$

$$L_\pi = \frac{4n_{1e}w^2}{3\lambda}$$



Self-imaging within an MMI coupler



Okamoto, Fundamentals of Optical Waveguides

Self-imaging: Input field is reproduced in single and multiple images along the propagation direction of the MMI section

$$\Phi(x,y,6L_\pi) = \Phi(x,y,0)\,\mathrm{e}^{-\mathrm{j}\,\beta_0\cdot 6L_\pi}$$

$$\Phi(x,y,3L_\pi) = \Phi(-x,y,0)\,\mathrm{e}^{-\mathrm{j}\,\beta_0\cdot 3L_\pi}$$

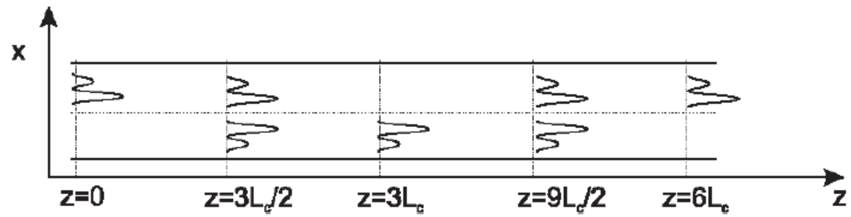
$$\Phi\left(x,y,\frac{3L_\pi}{2}\right) = \left[\frac{1-\mathrm{j}}{2}\Phi(x,y,0) + \frac{1+\mathrm{j}}{2}\Phi(-x,y,0)\right]\mathrm{e}^{-\mathrm{j}\,\beta_0\cdot\frac{3L_\pi}{2}}$$

… for a suitable excitation of modes at $z = 0$, see below (this does not hold for the situation depicted on the left!)

N images after propagation distance

$$z = \frac{3pL_\pi}{N}$$

where p and N are integers without common divider



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Further information on MMI couplers

Karlsruhe Institute of Technology

General self-imaging properties in $N \times N$ multimode interference couplers including phase relations

M. Bachmann, P. A. Besse, and H. Melchior

Self-imaging properties of generalized $N \times N$ multimode interference couplers are derived. Positions, amplitudes, and phases of the self-images are directly related to the lengths and widths of the coupler by solving the eigenmode superposition equation analytically for any arbitrary length. Devices of length $(M/N)\, 3L_c$, where M is the multiple occurrence of the N self-images, are analyzed in detail. The general formalism is applied to practical $N \times N$ couplers used in integrated optics, and simple phase relations are obtained.

Key words: Integrated optics, optical splitters and combiners, self-imaging.

1. Introduction

Splitting and combining of multiple optical beams is an important function in integrated optics. Multimode waveguides can be used to form multiple images. Ulrich and Ankele,[1] after a suggestion by Bryngdahl,[2] first demonstrated this effect in planar glass waveguides. Using ray optics, they successfully described the formation of multiple images at certain device lengths. The phases of the images, however, have not been given in a compact analytic expression. Ten years later, Niemeier and Ulrich[3] reported a

comparison with straight waveguides. Record low values for cross talk and imbalance have been reported. It has also been demonstrated that these values can be maintained fairly independently of polarization and over wide ranges of operation wavelengths[6] and operating temperatures.[5] Usually, 2×2 devices are short, well below 1 mm.[4–6] Even ultrashort (only 20–30-μm-long) 3-dB splitters have been reported.[10] Soldano *et al.* demonstrated good fabrication tolerances by optimizing the widths of the MMI couplers.[6]

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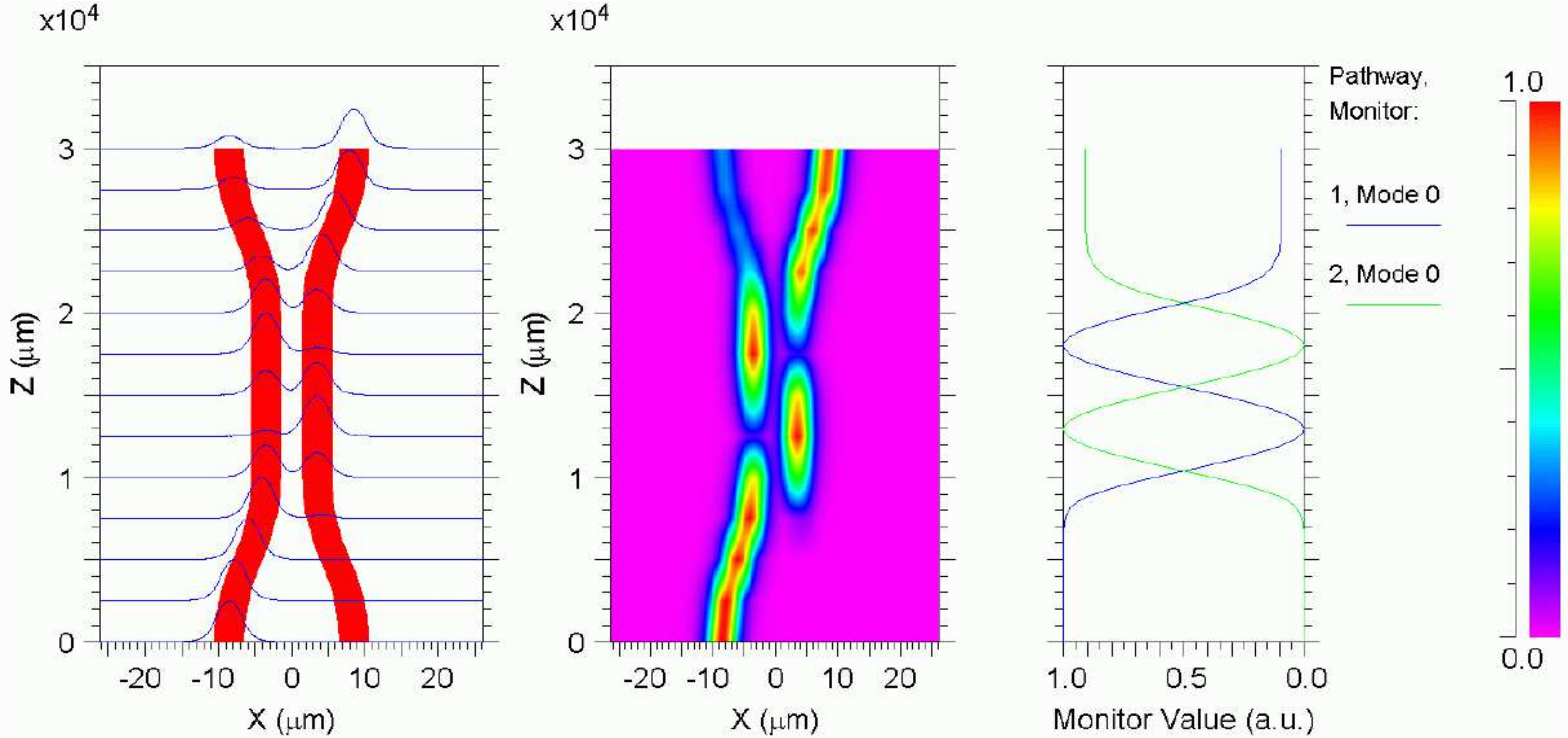
Directional coupler

Closely spaced waveguides running in parallel to each other
⇒ Evanescent tails of individual eigenmodes interact with neighboring waveguides
⇒ Oscillation of power between the two waveguides



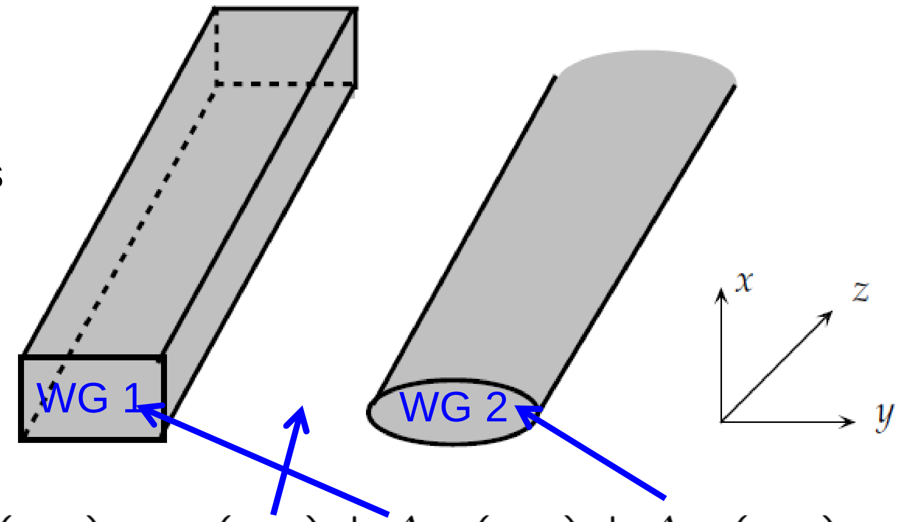
Analysis of directional couplers

Perturbation approach:

- Electromagnetic field represented by a superposition of modes of the indivudal waveguides with z-dependent mode amplitudes
- For simplicity: Consider guided modes only

$$\underline{\mathbf{E}}(x,y,z) = \sum_{\nu=1}^{2} A_\nu(z)\,\underline{\mathcal{E}}_\nu(x,y)\,\mathrm{e}^{-\mathrm{j}\,\beta_\nu z}$$

$$\underline{\mathbf{H}}(x,y,z) = \sum_{\nu=1}^{2} A_\nu(z)\,\underline{\mathcal{H}}_\nu(x,y)\,\mathrm{e}^{-\mathrm{j}\,\beta_\nu z}$$



$$\epsilon(x,y) = \epsilon_b(x,y) + \Delta\epsilon_1(x,y) + \Delta\epsilon_2(x,y)$$

Maxwell's equations for total structure:

$$\nabla \times \underline{\mathbf{E}}(x,y,z) = -\mathrm{j}\,\omega\mu_0 \underline{\mathbf{H}}(x,y,z)$$

$$\nabla \times \underline{\mathbf{H}}(x,y,z) = \mathrm{j}\,\omega\left(\epsilon_b(x,y) + \Delta\epsilon_1(x,y,z) + \Delta\epsilon_2(x,y,z)\right)\underline{\mathbf{E}}(x,y,z),$$

Note: $\nabla \times (\Phi \mathbf{F}) = \Phi(\nabla \times \mathbf{F}) + (\nabla\Phi \times \mathbf{F})$

Maxwell's equations of individual waveguides:

$$\nabla \times \left(\underline{\mathcal{E}}_\nu \exp(-\mathrm{j}\,\beta_\nu z)\right) = -\mathrm{j}\,\omega\mu_0\left(\underline{\mathcal{H}}_\nu \exp(-\mathrm{j}\,\beta_\nu z)\right),$$

$$\nabla \times \left(\underline{\mathcal{H}}_\nu \exp(-\mathrm{j}\,\beta_\nu z)\right) = \mathrm{j}\,\omega\left(\epsilon_b + \Delta\epsilon_\nu\right)\left(\underline{\mathcal{E}}_\nu \exp(-\mathrm{j}\,\beta_\nu z)\right).$$

Orthogonality relation for guided modes:

$$\frac{1}{4}\iint_{-\infty}^{\infty}\left(\underline{\mathcal{E}}_\nu(x,y) \times \underline{\mathcal{H}}_\mu^\star(x,y) + \underline{\mathcal{E}}_\mu^\star(x,y) \times \underline{\mathcal{H}}_\nu(x,y)\right)\cdot \mathbf{e}_z\,\mathrm{d}\,x\,\mathrm{d}\,y = \mathcal{P}_\mu \delta_{\nu\mu}$$

Analysis of directional couplers

Mode coupling equations:

$$\frac{\partial \underline{A}_1(z)}{\partial z} = -\mathrm{j}\,\kappa_{12}\underline{A}_2(z)\,\mathrm{e}^{-\mathrm{j}(\beta_2-\beta_1)z}$$

$$\frac{\partial \underline{A}_2(z)}{\partial z} = -\mathrm{j}\,\kappa_{21}\underline{A}_1(z)\,\mathrm{e}^{-\mathrm{j}(\beta_1-\beta_2)z}$$

where $$\kappa_{\nu\mu} = \frac{\omega}{4\mathcal{P}_\nu}\iint_{-\infty}^{\infty} \Delta\epsilon_\nu(x,y)\,\underline{\mathcal{E}}_\mu(x,y)\cdot\underline{\mathcal{E}}_\nu^\star(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y$$

Notes:

- Here, the mode amplitudes A(z) are dimensionless quantities
- $\mathscr{P}_\nu$ is used for power normalization of the mode fields

$$\mathcal{P}_\nu = \frac{1}{2}\iint_{-\infty}^{\infty} \mathrm{Re}\left\{\underline{\mathcal{E}}_\nu(x,y)\times\underline{\mathcal{H}}_\nu^\star(x,y)\right\}\cdot\mathrm{e}_z\,\mathrm{d}\,x\,\mathrm{d}\,y.$$

- The physical power P_ν is given by the mode amplitude and the power contained in the associated mode field:

$$P_\nu(z) = \left|\underline{A}_\nu(z)\right|^2\mathcal{P}_\nu$$

- In many cases, both waveguides have the same cross section and both mode fields are normalized to the same power. We may then simplify the mode coupling equations:

$$\beta_1 = \beta_2 = \beta \qquad \kappa_{12} = \kappa_{21}^\star$$

- Phases of mode fields can then be adjusted such that κ is real $\quad \kappa_{12} = \kappa_{21}^\star = \kappa$

Directional coupler

For identical waveguides (A_1 (0) = A_0; A_2(0) = 0):

$$\beta_1 = \beta_2 = \beta; \qquad \kappa_{12} = \kappa_{21} = \kappa$$

$$A_1(z) = A_0 \cos(\kappa z)$$
$$A_2(z) = -\mathrm{j}\, A_0 \sin(\kappa z)$$

i.e., power is oscillating back and forth between the waveguides.

Equal power in both waveguides for κ z = π/4:

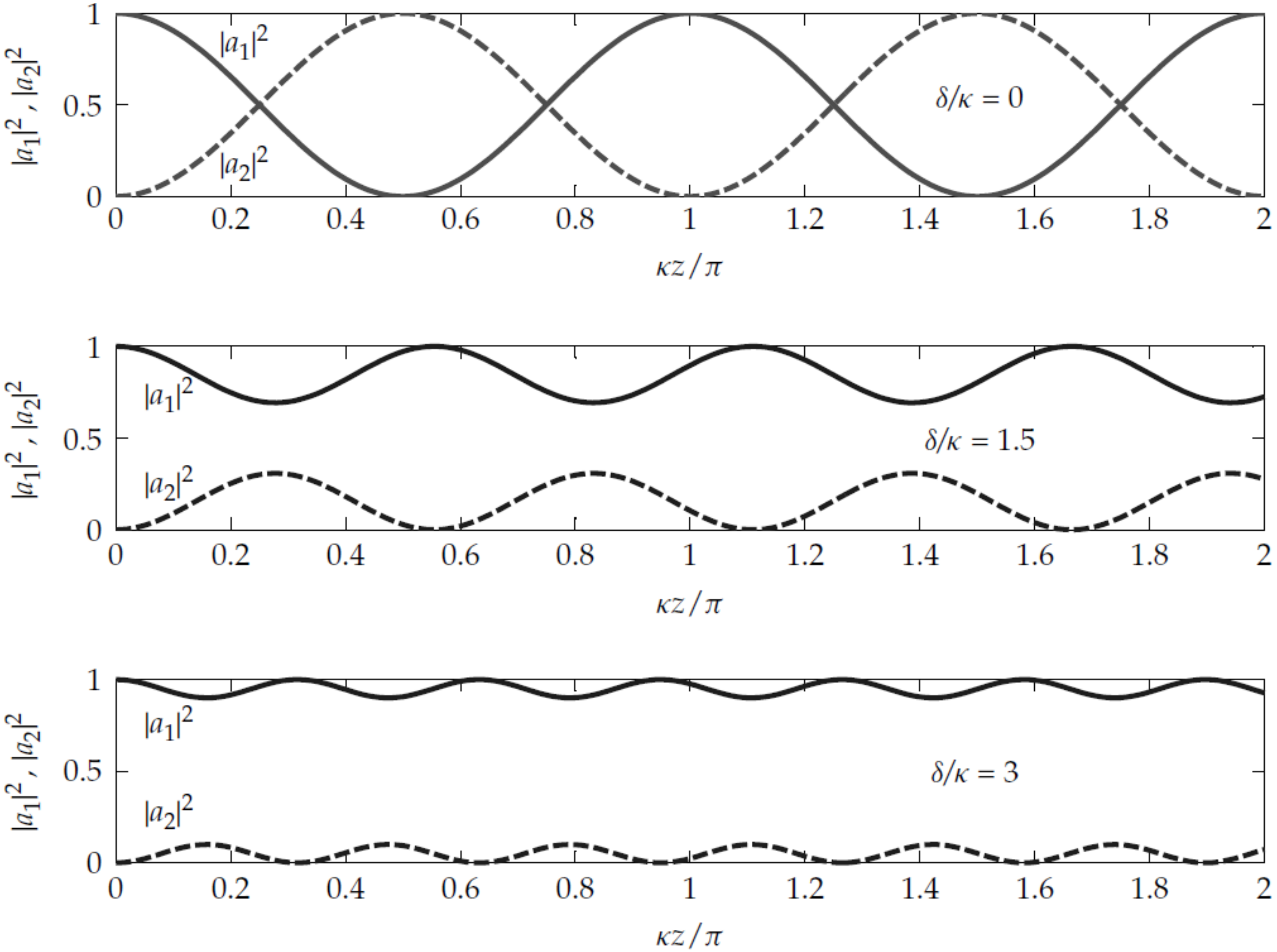
$$L_{3\mathrm{dB}} = \frac{\pi}{4\kappa}$$

General solutions for coupled waveguides:

$$A_1(z) = \left[\left(\cos(\sigma z) + \mathrm{j}\frac{\delta}{\sigma}\sin(\sigma z)\right) A_1(0) - \mathrm{j}\frac{\kappa}{\sigma}\sin(\sigma z)\, A_2(0)\right] \mathrm{e}^{-\mathrm{j}\,\delta z}$$
$$A_2(z) = \left[-\mathrm{j}\frac{\kappa}{\sigma}\sin(\sigma z)\, A_1(0) + \left(\cos(\sigma z) - \mathrm{j}\frac{\delta}{\sigma}\sin(\sigma z)\right) A_2(0)\right] \mathrm{e}^{\mathrm{j}\,\delta z}$$

where $\sigma^2 = \kappa^2 + \delta^2; \qquad \delta = \frac{\beta_2 - \beta_1}{2}$ Mismatch of propagation constant

Power oscillations in directional couplers



identical waveguide cross sections, no “detuning”

Strong “detuning”, incomplete power transfer

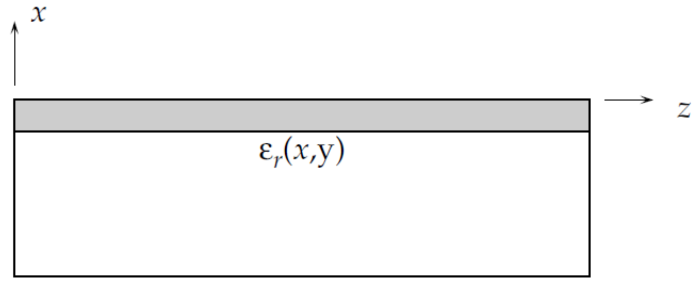
Chen, Foundations for guided-wave optics

Mechanical analogon: Coupled pendula
http://www.theorphys.science.ru.nl/people/fasolino/sub_java/pendula/laboratory-en.shtml

Waveguide gratings: Coupled-mode theory

Regular, z-independent waveguides:

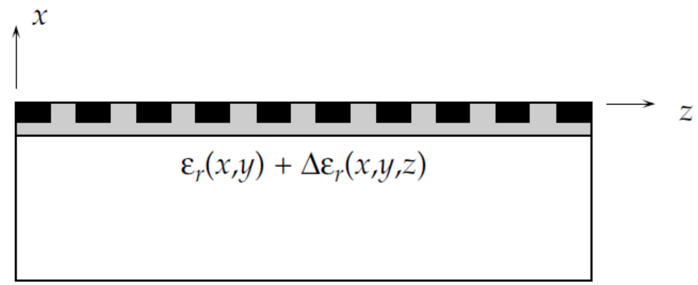
- Orthogonal eigenmodes propagate in an independent way
- Field can be written as a linear superposition of eigenmodes with constant mode amplitudes



Slightly irregular wavegeuides with small deviations from z-independent structure:

- Can be considered as perturbed *z*-independent structure:

$$\epsilon_p\,(x,y,z) = \epsilon\,(x,y) + \Delta\epsilon\,(x,y,z)$$



- Field can be approximated by modes of unperturbed structures with z-dependent mode amplitudes:

$$\underline{\mathbf{E}}(x,y,z) = \sum_{\nu} \underline{A}_{\nu}\,(z)\,\underline{\mathcal{E}}_{\nu}(x,y)\,\exp\left(-\mathrm{j}\,\beta_{\nu} z\right) + \sum_{\mu} \int_{\rho} \underline{A}_{\mu}\,(\rho,z)\,\underline{\mathcal{E}}_{\rho,\mu}(x,y)\,\exp\left(-\mathrm{j}\,\beta_{\mu}\,(\rho)\,z\right)\,\mathrm{d}\,\rho$$

$$\underline{\mathbf{H}}(x,y,z) = \sum_{\nu} \underline{A}_{\nu}\,(z)\,\underline{\mathcal{H}}_{\nu}(x,y)\,\exp\left(-\mathrm{j}\,\beta_{\nu} z\right) + \sum_{\mu} \int_{\rho} \underline{A}_{\mu}\,(\rho,z)\,\underline{\mathcal{H}}_{\rho,\mu}(x,y)\,\exp\left(-\mathrm{j}\,\beta_{\mu}\,(\rho)\,z\right)\,\mathrm{d}\,\rho$$

- The evolution of the mode amplitudes is dictated by the coupling of these modes due to the perturbation of the waveguide

Chen, Foundations for guided-wave optics

Coupled-mode equations for guided modes

Maxwell's curl equations for the perturbed waveguide structure:

$$\nabla \times \underline{\mathbf{E}}(x,y,z) = -\mathrm{j}\,\omega\mu_0 \underline{\mathbf{H}}(x,y,z)$$
$$\nabla \times \underline{\mathbf{H}}(x,y,z) = \mathrm{j}\,\omega\left(\epsilon\,(x,y) + \Delta\epsilon\,(x,y,z)\right)\underline{\mathbf{E}}(x,y,z),$$

Mode expansion (for simplicity, consider guided modes only):

$$\underline{\mathbf{E}}(x,y,z) = \sum_{\nu} \underline{A}_{\nu}\,(z)\,\underline{\mathcal{E}}_{\nu}(x,y)\,\exp\left(-\mathrm{j}\,\beta_{\nu}z\right)$$
$$\underline{\mathbf{H}}(x,y,z) = \sum_{\nu} \underline{A}_{\nu}\,(z)\,\underline{\mathcal{H}}_{\nu}(x,y)\,\exp\left(-\mathrm{j}\,\beta_{\nu}z\right)$$

Using the orthogonality relation, the **mode coupling equations** can be derived:

$$\frac{\partial \underline{A}_{\mu}\,(z)}{\partial z} = -\mathrm{j}\sum_{\nu} \kappa_{\mu\nu}\,(z)\,\underline{A}_{\nu}\,(z)\,\mathrm{e}^{-\mathrm{j}(\beta_{\nu}-\beta_{\mu})z}$$

where the **coupling coefficients** are given by:

$$\kappa_{\mu\nu} = \frac{\omega}{4\mathcal{P}_{\mu}}\iint_{-\infty}^{\infty} \Delta\epsilon\,(x,y,z)\,\underline{\mathcal{E}}_{\nu}(x,y)\cdot\underline{\mathcal{E}}_{\mu}^{\star}(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y$$

Interpretation: The dielectric perturbation is "weighted" by the electric fields of the modes. Coupling is most effective if perturbations occur in regions where the E-fields are strong!

Coupled-mode equations for guided modes

Note:

- The quantity

$$\mathcal{P}_\mu = \frac{1}{2}\iint_{-\infty}^{\infty} \mathrm{Re}\left\{\underline{\mathcal{E}}_\mu(x,y) \times \underline{\mathcal{H}}_\mu^\star(x,y)\right\} \cdot \mathrm{e}_z \, \mathrm{d}\,x \, \mathrm{d}\,y$$

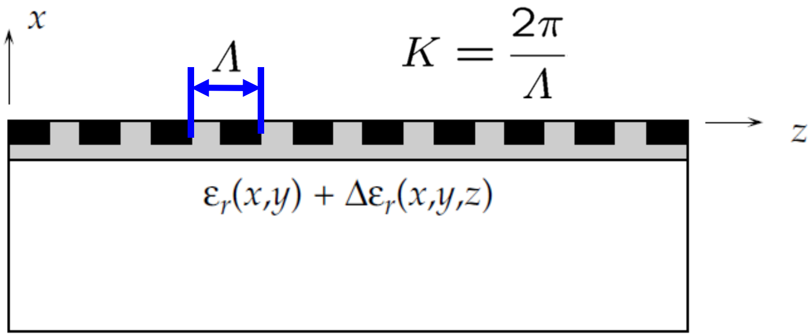
  gives the power that is contained in the mode field representations. Note that $\mathscr{P}_\mu < 0$ for modes that propagate to the left. The physical power flux is given by $P = |A_\mu|^2 \mathscr{P}_\mu$

- If all modes have the same power normalization ($\mathscr{P}_1 = \mathscr{P}_2 = \ldots$) and if $\Delta\epsilon$ is real, then the coupling coefficients obey the following symmetry relations:

$$\kappa_{\nu\mu} = \begin{cases} -\kappa_{\mu\nu}^\star & \text{for counterpropagating modes} \\ \kappa_{\mu\nu}^\star & \text{for co-propagating modes} \end{cases}$$

- The phases of two mode fields ν and μ can be adjusted such that $\kappa_{\nu\mu}$ is real

Mode coupling by periodic perturbations



Taylor expansion of periodic perturbation:

$$\Delta\epsilon(x,y,z) = \sum_{q=-\infty}^{\infty} \Delta\epsilon_q(x,y)\,\mathrm{e}^{-\mathrm{j}\,qKz}$$

Mode-coupling equation:

$$\frac{\partial \underline{A}_\mu(z)}{\partial z} = -\mathrm{j}\sum_{\nu}\sum_{q}\kappa_{\mu\nu,q}\underline{A}_\nu(z)\,\mathrm{e}^{-\mathrm{j}(\beta_\nu-\beta_\mu+qK)z}$$

where

$$\kappa_{\mu\nu,q} = \frac{\omega}{4\mathcal{P}_\mu}\iint_{-\infty}^{\infty}\Delta\epsilon_q(x,y)\,\underline{\mathcal{E}}_\nu(x,y)\cdot\underline{\mathcal{E}}_\mu^\star(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y$$

Interpretation: Two modes with propagation constants β_ν and β_μ are most effectively coupled by the spatial frequency component *qK* that corresponds to the difference β_ν - β_μ

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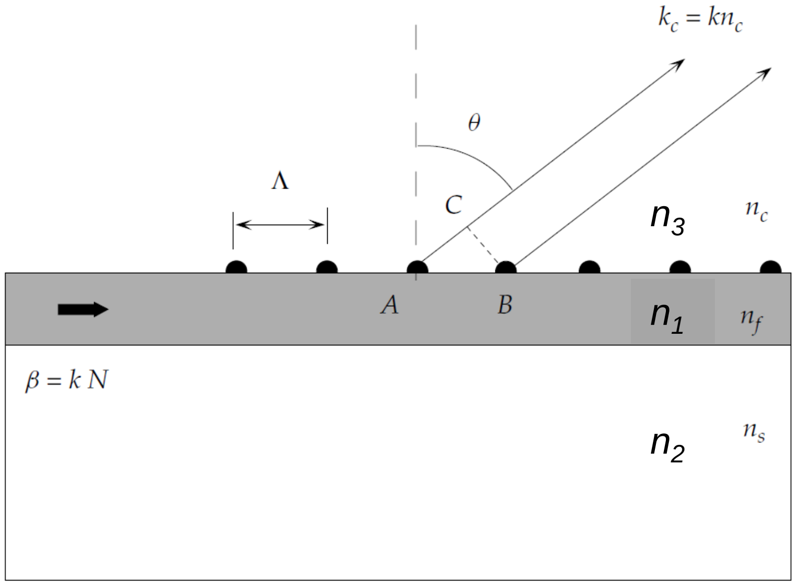
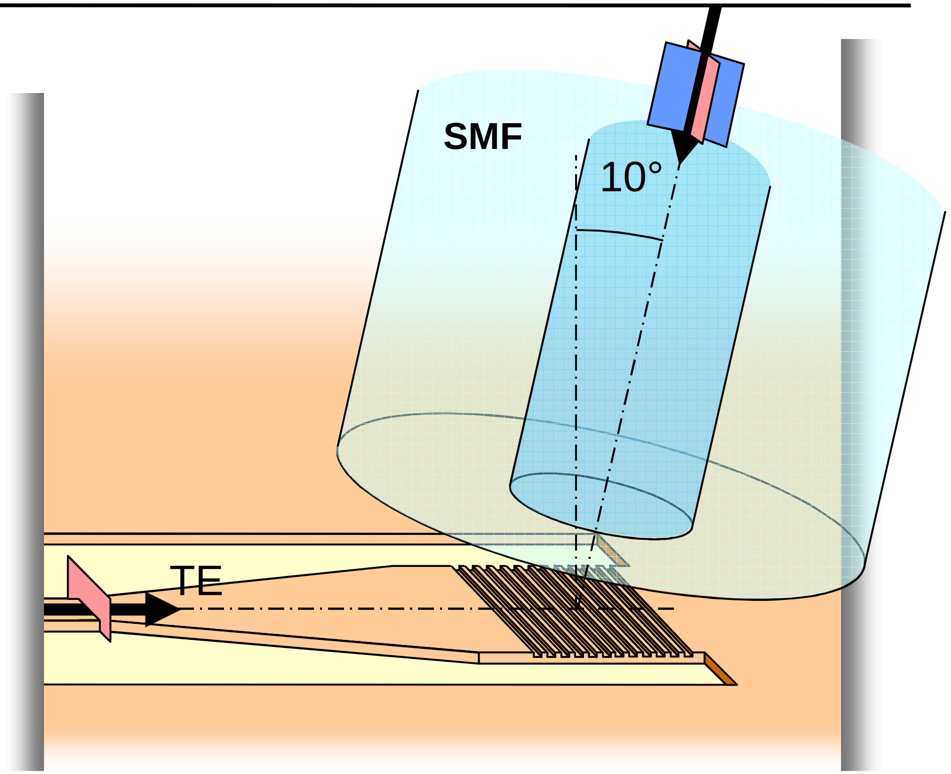
Example: Grating-assisted fiber-chip coupling





Coupling of guided mode to radiation modes by periodic perturbation:

$$\beta_{\text{rad}} = n_3 k_0 \sin\Theta$$
$$\beta_{\text{guid}} = n_e k_0$$

First-order grating (q = -1):

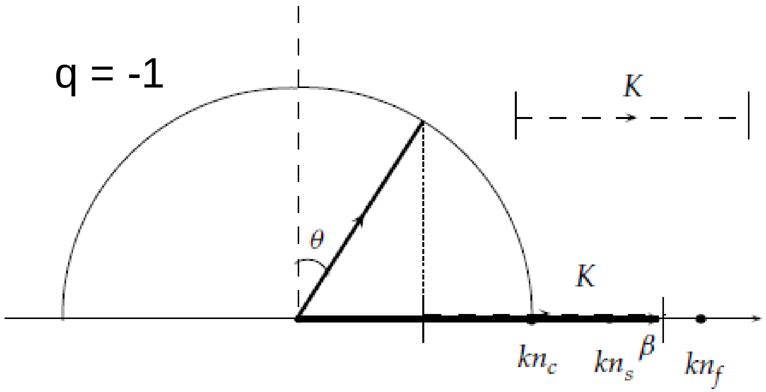
$$K = \beta_{\text{guid}} - \beta_{\text{rad}}$$
$$\Lambda = \frac{\lambda}{n_e - n_3 \sin\Theta}$$

Chen, Foundations for guided-wave optics

Example: Grating-assisted coupling and mode conversion

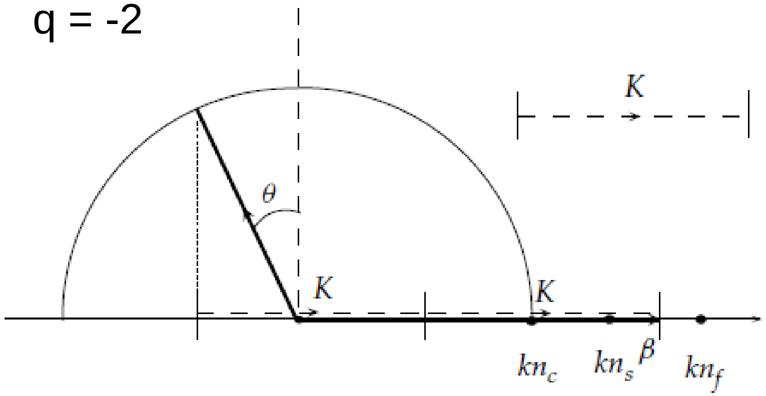
Higher-order diffraction in Grating-assisted fiber-chip couplers:

q = -1



(a)

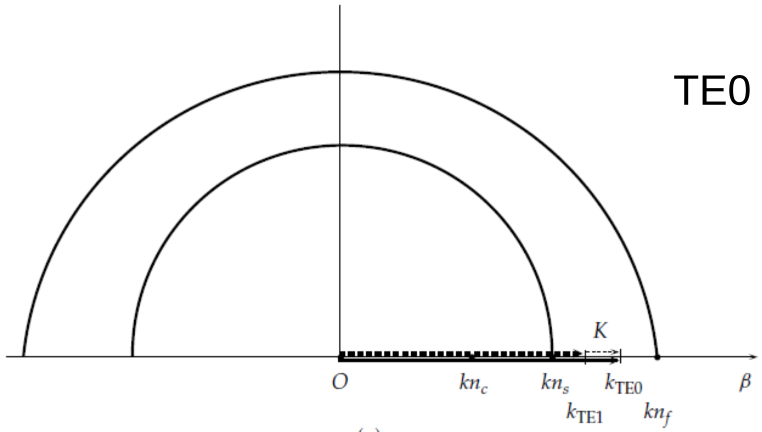
q = -2



(b)

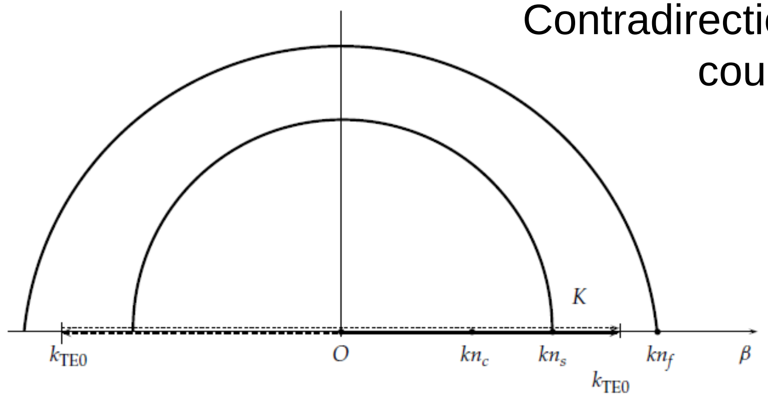
Grating-assisted mode conversion:

TE0 -> TE1



(a)

Contradirectional coupling



(b)

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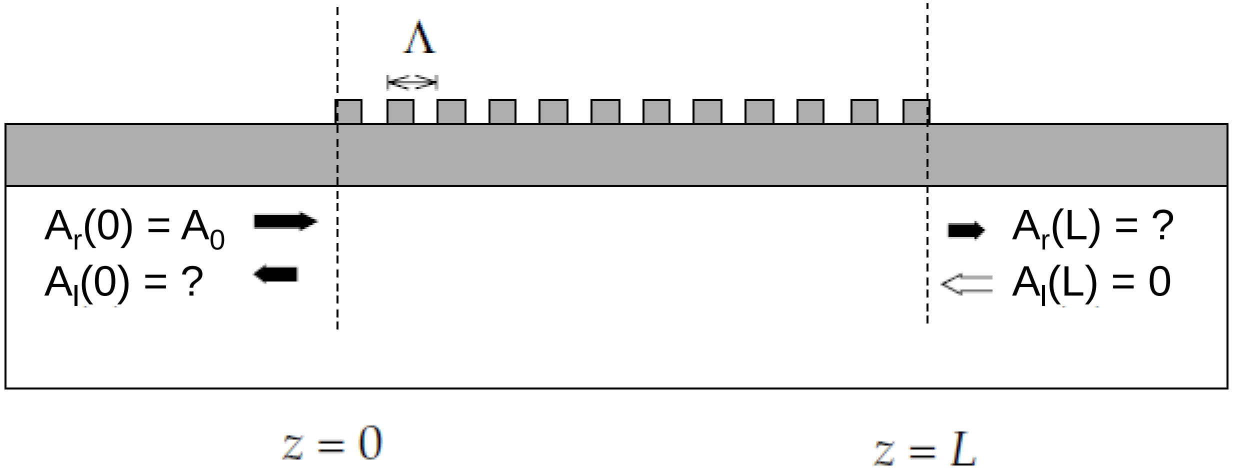
Example: Contra-directional coupling and Waveguide Bragg gratings



Consider two counter-propagating but otherwise identical modes:

$$\beta_r = -\beta_l = \beta$$
$$\mathcal{P}_r = -\mathcal{P}_l$$
$$\kappa_{rl} = -\kappa_{lr}^{\star} = \kappa$$

Coupled-mode equations for perturbation $\Delta\epsilon$(x,y,z) with zero mean value along z:

$$\frac{\partial \underline{A}_l(z)}{\partial z} = \mathrm{j}\,\kappa \underline{A}_r(z)\,\mathrm{e}^{-\mathrm{j}\,2\delta z}$$
$$\frac{\partial \underline{A}_r(z)}{\partial z} = -\mathrm{j}\,\kappa \underline{A}_l(z)\,\mathrm{e}^{\mathrm{j}\,2\delta z}$$

where $\delta = \beta - \frac{K}{2}$ Detuning (Bragg) parameter

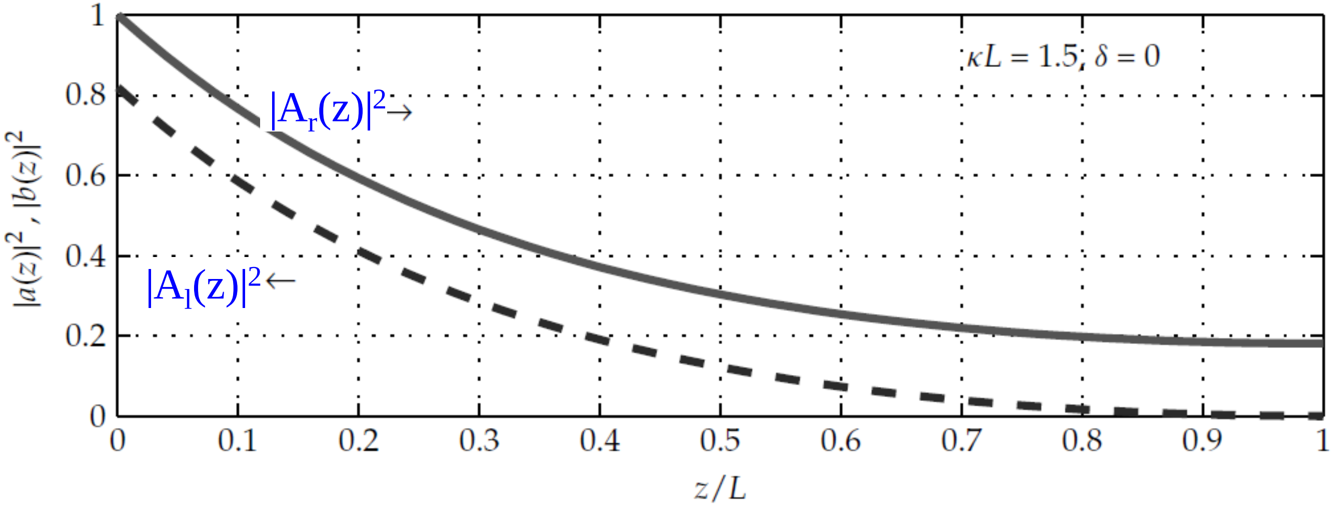
General solution:

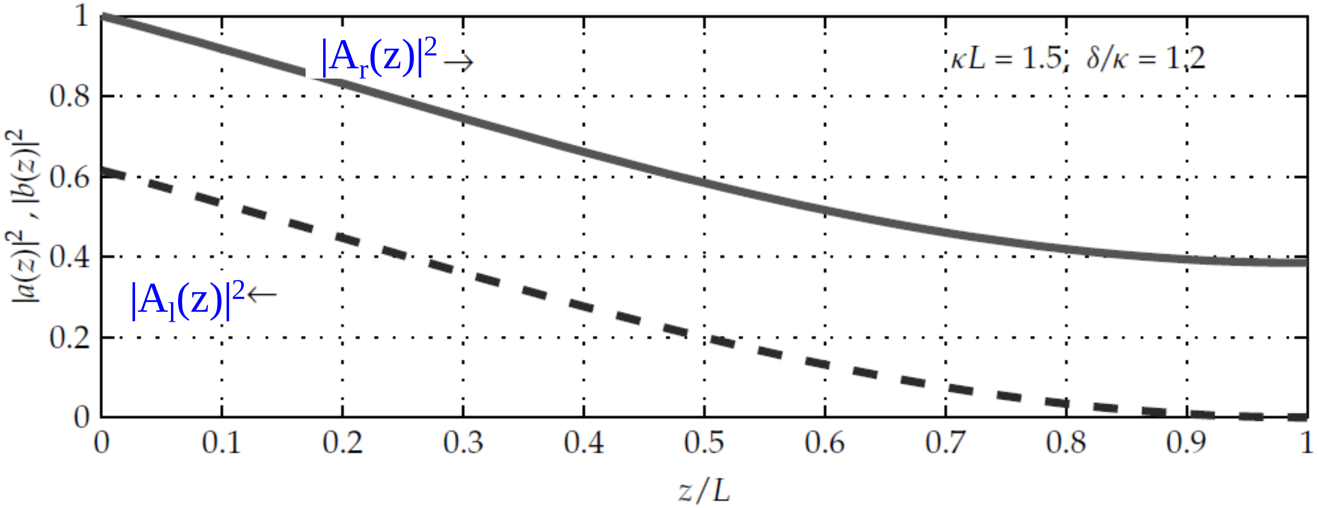
$$\underline{A}_r(z) = A_0 \frac{\sigma \cosh(\sigma(z-L)) - \mathrm{j}\,\delta \sinh(\sigma(L-z))}{\sigma \cosh(\sigma L) + \mathrm{j}\,\delta \sinh(\sigma L)}$$
$$\underline{A}_l(z) = A_0 \frac{\mathrm{j}\,\kappa \sinh(\sigma(z-L))}{\sigma \cosh(\sigma L) + \mathrm{j}\,\delta \sinh(\sigma L)}$$

where $\sigma^2 = \kappa^2 - \delta^2$

Chen, Foundations for guided-wave optics

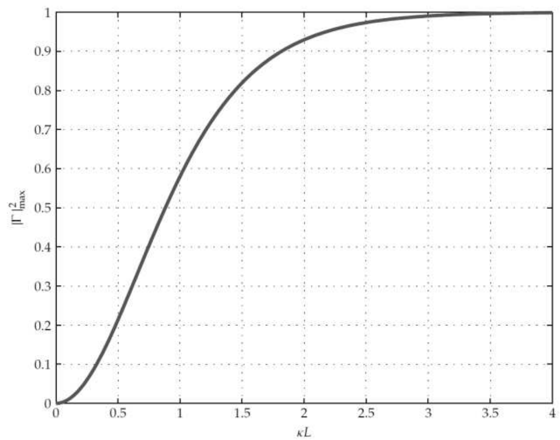
Waveguide Bragg gratings: Transmission and reflection





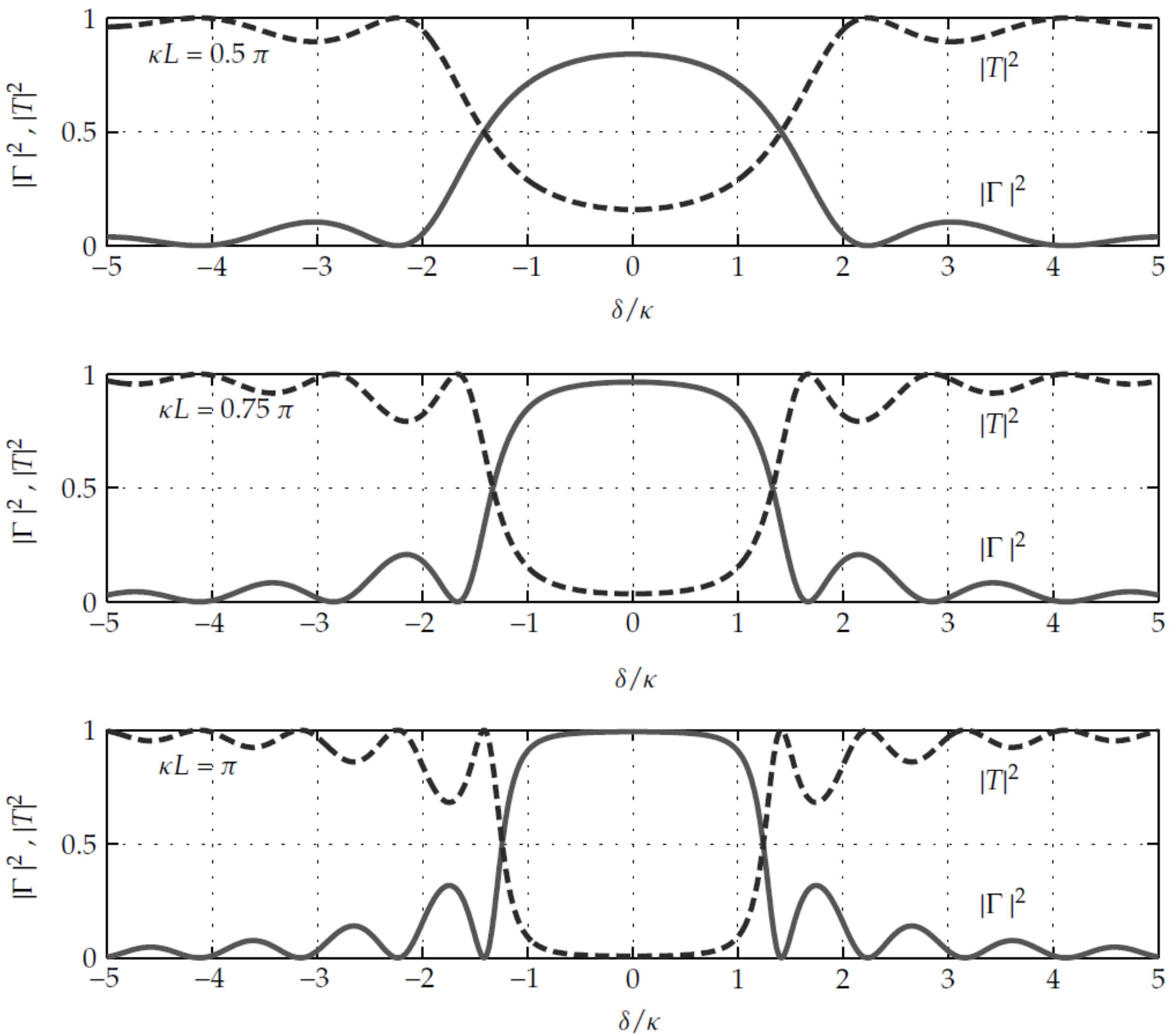
Peak power reflectance for $\delta = 0$:

$$|\Gamma|^2_{\max} = \frac{|\underline{A}_l(0)|^2}{|\underline{A}_r(0)|^2} = \tanh^2(\sigma L)$$



Chen, Foundations for guided-wave optics

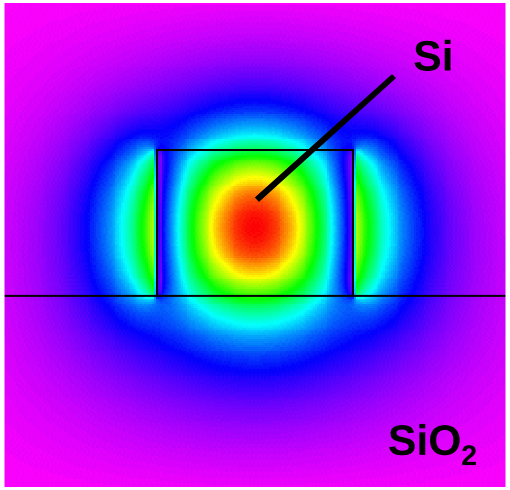
Waveguide Bragg gratings: Transmission and reflection



Chen, Foundations for guided-wave optics

Material absorption and modal loss

Assumption: Waveguide core is affected by absorption loss, i.e., the complex refractive index has a nonzero imaginary part. This can be interpreted as a perturbation of the refractive index profile:



$$\underline{\epsilon} = \epsilon_0 \underline{n}^2 = \epsilon_0 \left(n - \mathrm{j}\, n_i\right)^2 \approx \epsilon_0 n^2 - \mathrm{j}\, \epsilon_0\, 2 n n_i$$

$$\Delta \underline{\epsilon} = -\mathrm{j}\, \epsilon_0\, 2 n n_i$$

By formally applying the coupled mode theory to the fundamental mode, we obtain the power attenuation along the waveguide:

$$\frac{\partial A_\nu}{\partial z} = -\frac{1}{2}\alpha_\nu A_\nu \qquad\qquad |A_\nu(z)|^2 = |A_\nu(0)|^2\, \mathrm{e}^{-\alpha_\nu z}$$

where: $\alpha_\nu = \Gamma_\nu \alpha_m$ Modal loss coefficient

$\alpha_m = 2 k_0 n_i$ Material loss coefficient

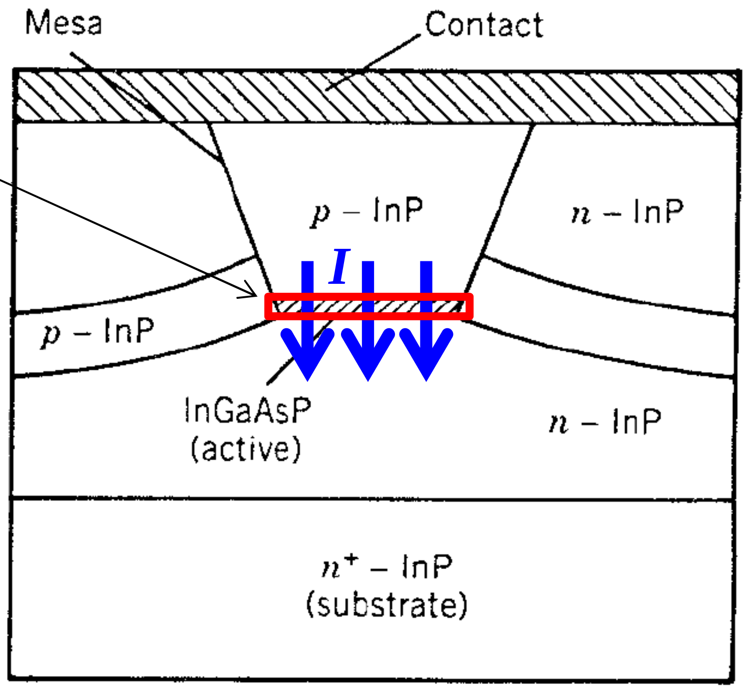
$$\Gamma_\nu = \frac{\omega \epsilon_0 n_{\text{core}}}{2 k_0 \mathcal{P}_\nu} \iint_{\text{core}} |\underline{\mathcal{E}}_\nu(x,y)|^2 \,\mathrm{d}\,x\,\mathrm{d}\,y$$

$$\approx \frac{\iint_{\text{core}} |\underline{\Psi}_\nu(x,y)|^2 \,\mathrm{d}\,x\,\mathrm{d}\,y}{\iint_{-\infty}^{\infty} |\underline{\Psi}_\nu(x,y)|^2 \,\mathrm{d}\,x\,\mathrm{d}\,y}$$

Field confinement factor (“fraction of optical power that propagates in the waveguide core”)

Modal gain in an active optical waveguide:

Active waveguide core
= gain region (g.r.)



$$\frac{\partial A_\nu}{\partial z} = \frac{1}{2} g_\nu A_\nu$$

$$|A_\nu(z)|^2 = |A_\nu(z)|^2 \, \mathrm{e}^{g_\nu z}$$

where:

$g_\nu = \Gamma_\nu g_m$ Modal gain coefficient

g_m Material gain coefficient

$$\Gamma_\nu = \frac{\omega \epsilon_0 n_{\mathrm{gr}}}{2 k_0 \mathcal{P}_\nu} \iint_{\mathrm{g.r.}} |\underline{\mathcal{E}}_\nu(x,y)|^2 \, \mathrm{d}\,x \, \mathrm{d}\,y$$

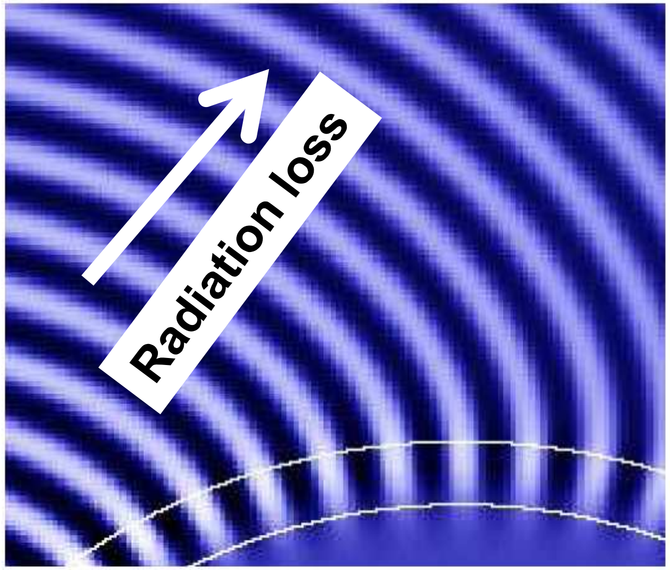
$$\approx \frac{\iint_{\mathrm{g.r.}} |\underline{\Psi}_\nu(x,y)|^2 \, \mathrm{d}\,x \, \mathrm{d}\,y}{\iint_{-\infty}^{\infty} |\underline{\Psi}_\nu(x,y)|^2 \, \mathrm{d}\,x \, \mathrm{d}\,y}$$

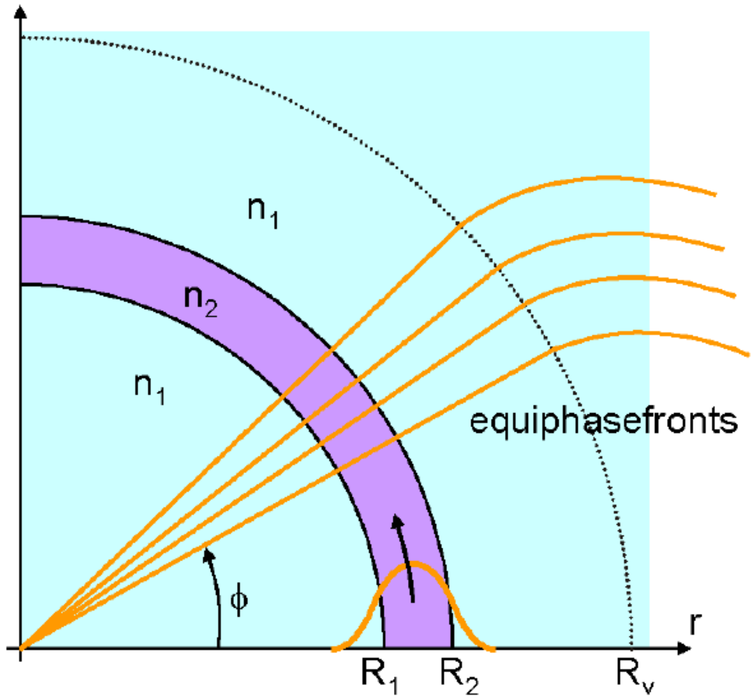
Field confinement to the gain region ("g.r.")

Bent waveguides

Radiation loss of bent waveguides

The phase front of the guided mode is rotating around the center of the bend. Because the group velocity of the phase fronts cannot exceed the speed of light (c/n), the phase fronts bend and cause radiation. The radiation loss increases exponentially with decreasing bend radius.





Analysis of waveguide bends:

- Assume low-index contrast waveguide structure and use scalar approximation.
- Reduce the problem to two dimensions by applying the effective index method.
- Use polar coordinates (r,φ) and consider propagation in azimuthal direction.

Differential operators in cylindrical coordinates…

Analysis of bent waveguides

Scalar Helmholtz equation in polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \underline{\Psi}(r,\varphi)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\underline{\Psi}(r,\varphi)}{\partial\varphi^2}+k_0^2n^2(r)\,\underline{\Psi}(r,\varphi)=0$$

Separation ansatz:

$$\underline{\Psi}(r,\varphi)=g(r)\,h(\varphi)$$

$$\frac{r^2}{g(r)}\frac{\partial^2 g(r)}{\partial r^2}+\frac{r}{g(r)}\frac{\partial g(r)}{\partial r}+k_0^2n^2(r)\,r^2=-\frac{1}{h(\varphi)}\frac{\partial^2 h(\varphi)}{\partial\varphi^2}$$

Azimuthal dependence: $h(\varphi)=C_1\,\mathrm{e}^{\pm\mathrm{j}\,\beta_\varphi\varphi}$

Remaining equation for radial dependence:

$$r^2\frac{\partial^2 g(r)}{\partial r^2}+r\frac{\partial g(r)}{\partial r}+\left(k_0^2n^2(r)\,r^2-\beta_\varphi^2\right)g(r)=0$$

Coordinate transformation from r to u:

$$r=R_t\,\mathrm{e}^{\frac{u}{R_t}}\qquad u=R_t\ln\left(\frac{r}{R_t}\right)\qquad u,r>0$$

$$g(r)=\overline{g}(u(r))$$

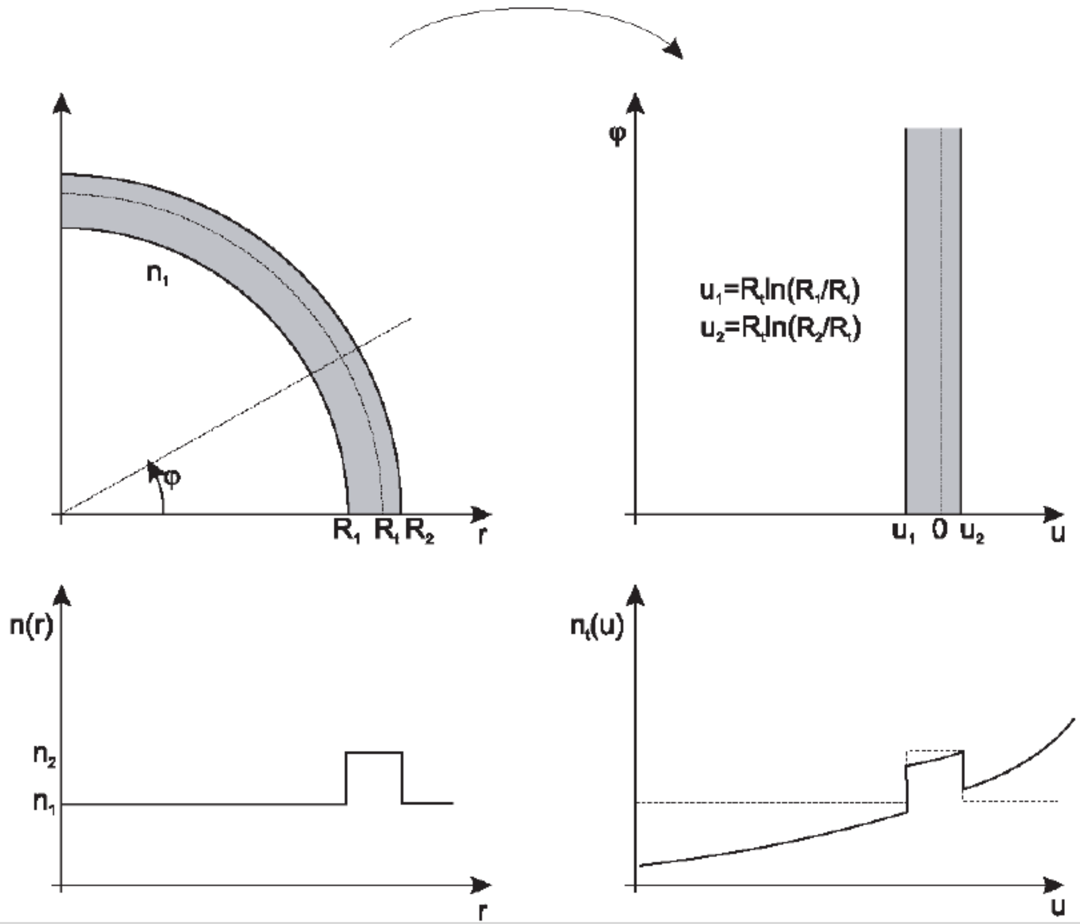
$$n(r)=\overline{n}(u(r))$$

Analysis of bent waveguides

Transformed Helmholtz equation: $$\frac{\partial^2 \overline{g}(u)}{\partial u^2} + \left(k_0^2 n_t^2(u) - \beta_t^2\right)\overline{g}(u) = 0$$

where $$n_t(u) = n\left(R_t\,\mathrm{e}^{\frac{u}{R_t}}\right)\mathrm{e}^{\frac{u}{R_t}} \qquad \beta_t = \frac{\beta_\varphi}{R_t}$$

$\Rightarrow$ The transformed Helmholtz equation in the (u,φ)-coordinate system has exactly the same form as in a Cartesian coordinate system if the refractive index profile is replaced by the transformed profile. Propagation in bent waveguides can therefore be calculated by solving the equivalent straight waveguide.



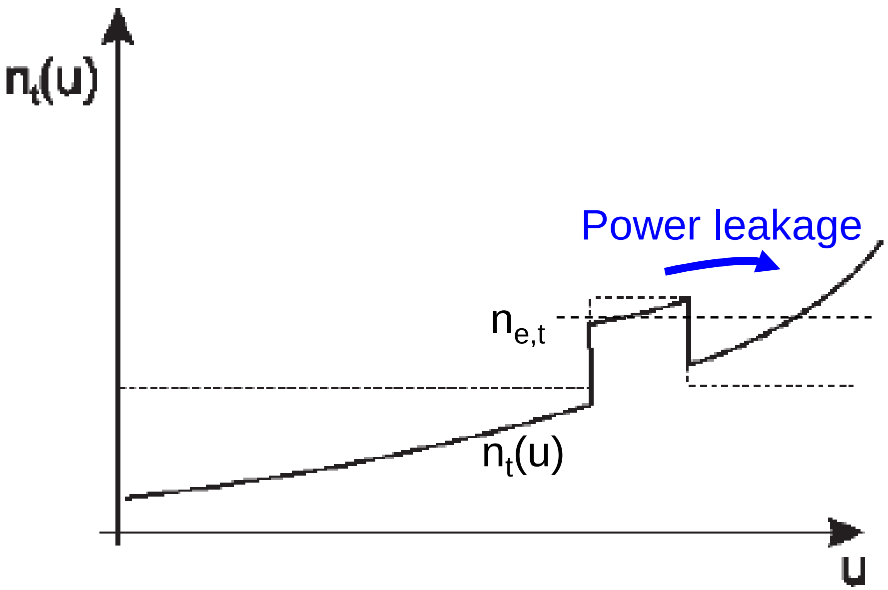
R. Baets, „Dielectric waveguides“, Lecture notes

Analysis of bent waveguides - discussion

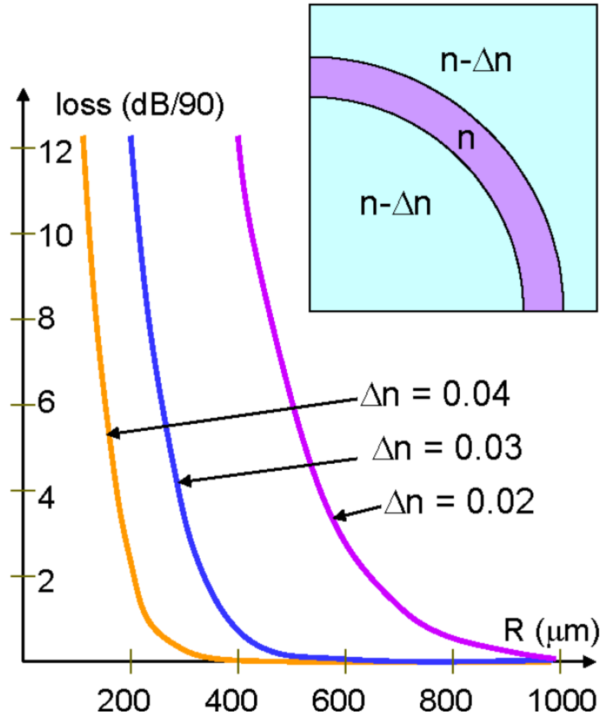
Power leakage will occur from the guided mode of the transformed waveguide (effective index $n_{e,t}$) to the region defined by:

$$\beta_t = n_{e,t} k_0 < n_t(u)\, k_0 \;\Leftrightarrow\; n_t(u) > n_{e,t}$$

This can be interpreted as „tunneling“ of photons through the „potential barrier“ defined by the curvature and the index contrast. Decreasing the radius of curvature makes the potential barrier narrower, and power leakage increases exponentially.

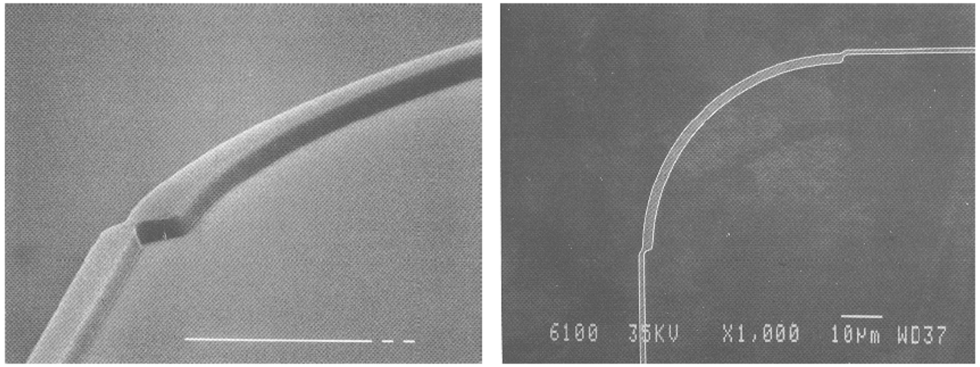


R. Baets, „Dielectric waveguides“, lecture notes



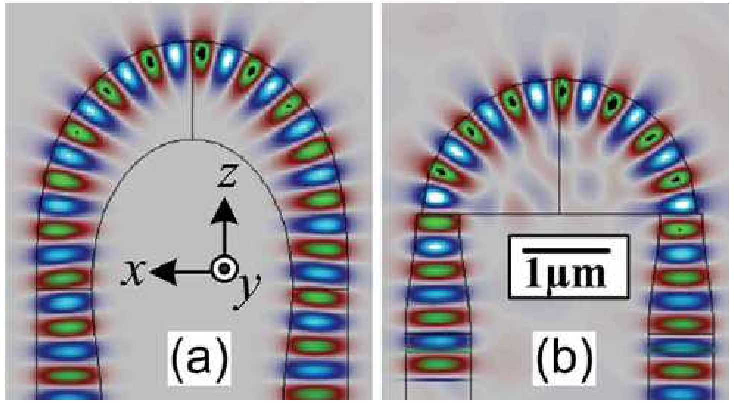
Analysis of bent waveguides - discussion

- The mode field will be concentrated in the region with the highest index, i.e., it will be „pressed“ towards the outer side of the bend. This leads to adaptation losses at the transition between the straight and the bent section. Power loss can be mitigated by a lateral offset at the transition.

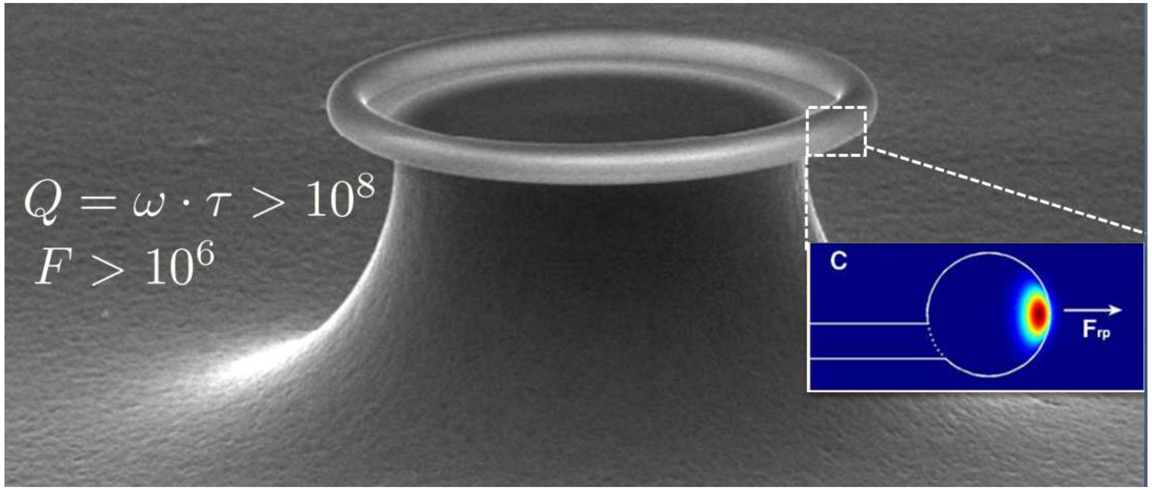


R. Baets, „Dielectric waveguides“, lecture notes

- For wide waveguides and strong curvature, the mode will only be guided by the outer contour of the waveguide, and the inner contour will not play a role. These modes are called whispering gallery modes.

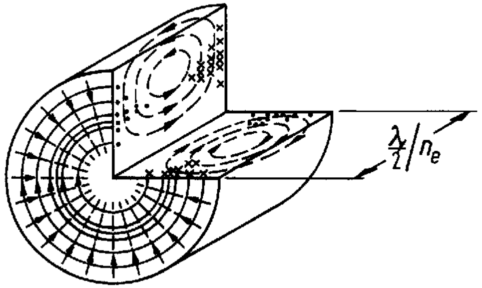


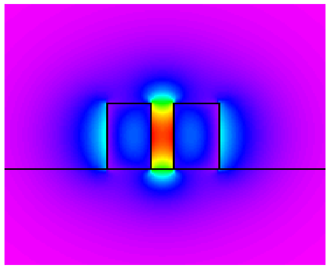
Koos *et al.*, IEEE Photon. Technol. Lett., Vol. 19, pp. 819-821 (2007)

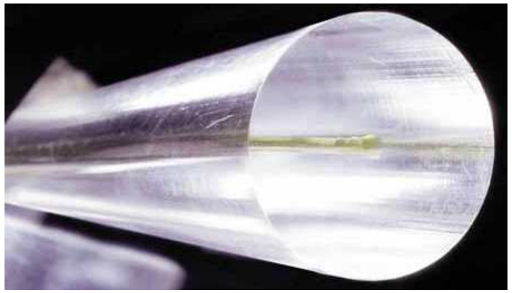


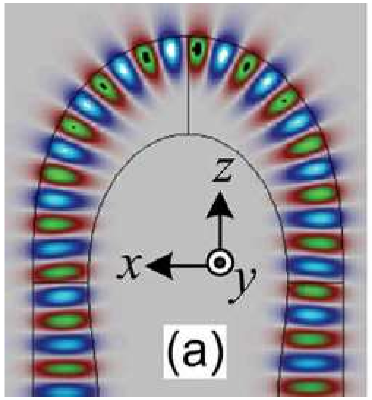
Del‘Haye *et al.*, Nature Dec. 20, 2007

Optical Waveguides and Fibers
- Summary -









Summary I

Fundamentals of Wave Propagation

- Dielectric polarization and susceptibility, complex refractive index
- Causality and Kramers-Kronig relation
- Absorption and material dispersion
- Lorentz oscillator model of bound charges, refractive index and absorption
- Sellmeier equations
- Drude mode for conductive media
- Signal propagation in dispersive media, group delay, group velocity, group refractive index, group velocity dispersion
- Material dispersion of fused silica

Dielectric slab waveguides

- Plane dielectric boundary: Reflection and transmission for TE and TM-polarization, power transmission and reflection, total internal reflection, field distribution
- Slab waveguide: Formation of guided modes, lateral self-consistence, interpretation of normalized waveguide parameters, graphical solution, TE and TM modes, field patterns of different guided modes
- Procedure for calculating for TE and TM modes
- Signal propagation in dispersive waveguides, effective group refractive index, dispersive effects, intermodal dispersion, chromatic dispersion, material dispersion, waveguide dispersion, “engineering” waveguide dispersion
- Formation of surface plasmon polaritons (SPP), special properties of SPP

Summary II

Planar integrated waveguides

- Different types of integrated waveguides
- Marcatili method: Basic idea, underlying assumptions, limitations
- E_x and E_y modes of a channel waveguide
- Effective index method: Basic idea, underlying assumptions, limitations
- Numerical mode solvers: General procedure, basic idea of finite difference mode solvers, termination of computational domain, sources of errors
- Integrated waveguide technologies: Glass waveguides, fabrication of waveguides based on ion exchange, proton exchange, polymer waveguides and absorption properties, silicon-based waveguides

Optical fibers

- General properties of fused-silica fibers: Fiber loss, transmission windows
- Solution procedure for step-index fibers (no derivation!), separation ansatz, qualitative dependence of fields on radial and azimuthal coordinates, “physically meaningful solutions”, fundamental mode, single-mode condition
- Hybrid modes and LP-modes, mode field nomenclature for LP modes
- Basic procedure to estimate the number of guided modes
- Graded-index fiber (no derivation, no formulae!), Gauss-Laguerre mode designation,
- Fiber materials and technologies: “Glass fibers”, fused silica fibers, fiber fabrication, polymer fibers, microstructured fibers

Summary III

- Fiber losses: Sources of loss, loss minimization
- Signal propagation in dispersive fibers, chromatic dispersion, waveguide dispersion, material dispersion, quantitative analysis of chromatic dispersion, slowly varying envelope approximation and retarded time frame, propagation of chirped Gaussian impulse
- Limitations of dispersive broadening on data rate
- Dispersion characteristics of single-mode fibers, CSF, DSF, DCF, dispersion engineering, dispersion compensation

Waveguide-based devices

- Mode expansion method: Guided modes and radiation modes, completeness and orthogonality of mode sets
- Coupling efficiency: Basic idea of analysis (no derivation!)
- Multi-mode interference coupler (MMI): Basic idea of analysis of self imaging properties (no derivation!)
- Directional couplers: Basic idea, coupling of power between parallel waveguides, “mechanical analogon”
- Waveguide gratings: Basic idea, mode coupling by periodic perturbations, grating-assisted fiber-chip coupling, waveguide Bragg gratings, co-directional coupling
- Absorption and gain in optical waveguides: Basic idea of analysis, field confinement factor
- Bent waveguides: Basic idea of analysis, transformation of index profile, power leakage, whispering-gallery modes